Mathematics

G. N. YAKOVLEV

1961

SovietRxiv

Abstract

Full Text

Mathematics

G. N. YAKOVLEV

BOUNDARY PROPERTIES OF FUNCTIONS OF THE CLASS $W_p^{(l)}$ ON DOMAINS WITH CORNER POINTS

(Presented by Academician M. A. Lavrent' ev, 18 IV 1961)

In the present note we study the properties of boundary values of functions of the class $W_p^{(l)}$, l a natural number, on plane domains with piecewise smooth boundary. The results obtained make it possible, in particular, to establish for such domains necessary and sufficient conditions for the solvability of the first boundary-value problem for the polyharmonic equation $\Delta^m u = 0$ in the class of functions with finite energy integral.

By $W_p^{(l)}$, l a natural number, $1 < p < \infty$, we shall denote the usual Sobolev class of functions ([1]). We shall say that $f(x)$ belongs to the class $W_p^{(l+r)}(a,b)$, $0 < r < 1$, $1 < p < \infty$, if $f(x) \in W_p^{(l)}(a,b)$ and the integral

$$A[f^{(l)};(a,b)] = \int_a^b \int_a^b \frac{|f^{(l)}(x_1) - f^{(l)}(x_2)|^p}{|x_1 - x_2|^{pr+1}}\, dx_1\, dx_2 < +\infty.$$

By the norm in the class $W_p^{(l+r)}(a,b)$ we shall mean the expression

$$\|f\|_{W_p^{(l+r)}(a,b)} = \|f\|_{W_p^{(l)}(a,b)} + \{A[f^{(l)};(a,b)]\}^{1/p}.$$

Such classes of functions, or special cases of these classes, have been considered in the works ([2–7]), etc.

1. In the first part of the present note we shall give some theorems on "matching" conditions for functions of the class $W_p^{(r)}$, $0 < r < 1$. For the classes $H_p^{(r)}$ analogous results were obtained by S. M. Nikol' skii ([8,9]).

Let $\psi_1(t)$ and $\psi_2(t)$ be some increasing nonnegative functions satisfying the conditions:

$$\psi_i(0) = 0, \qquad \lim_{t\to\infty} \psi_i(t) = \infty, \qquad 0 < C_1 < \psi_i'(t) < C_2 < \infty, \qquad i = 1, 2,$$

where C_1 and C_2 are some constants. Introduce the notation

$$I[f;\psi_1,\psi_2] = \int_0^\infty \frac{|f(-\psi_1(t)) - f(\psi_2(t))|^p}{t^{pr}}\,dt.$$

Then the following theorem holds:

Theorem 1. For any functions ψ_1 and ψ_2 defined above and any $f(x) \in L_p(-\infty,+\infty)$, the inequalities

$$C_3 A[f;(-\infty,+\infty)] \le A[f;(-\infty,0)] + A[f;(0,+\infty)] + I[f;\psi_1,\psi_2] \le C_4 A[f;(-\infty,+\infty)],$$

hold, where C_3 and C_4 are some positive constants depending on p, r and on the functions ψ_1, ψ_2, but not depending on the function f.

The proof of this theorem is carried out by direct calculation.

Theorem 2. Let $f(x) \in W_p^{(r)}(0,\infty)$. Then:

$$1)\quad \int_0^\infty x^{-pr}|f(x)|^p\,dx \le C_5 A[f;(0,\infty)], \quad \text{if } pr < 1;$$

$$2)\quad \int_0^\infty x^{-pr}|f(x) - f(0)|^p\,dx \le C_6 A[f;(0,\infty)], \quad \text{if } pr > 1,$$

where C_5 and C_6 are positive constants depending only on p and r.

In the proof of this theorem the following two identities are used:

$$f(x) = \frac{1}{x}\int_x^{2x} [f(x) - f(t)]\,dt + \frac{1}{x}\int_x^{2x} f(t)\,dt,$$

$$f(x) = \frac{1}{x}\int_0^{x} [f(x) - f(t)]\,dt + \frac{1}{x}\int_0^{x} f(t)\,dt$$

and the density of the set of sufficiently smooth functions in the class $W_p^{(r)}$.

From the theorems given above there follows immediately the following corollary:

Corollary. Let $f(x) \in W_p^{(r)}(0,\infty)$ and $f(x) \in W_p^{(r)}(-\infty,0)$. Then:

1) if $pr < 1$, then $f(x) \in W_p^{(r)}(-\infty,+\infty)$;

2) if $pr > 1$, $f(-0) = f(+0)$, then $f(x) \in W_p^{(r)}(-\infty,+\infty)$.

Moreover, in both cases the inequality holds:

$$A[f;(-\infty,+\infty)] \leq C_7\{A[f;(-\infty,0)] + A[f;(0,+\infty)]\},$$

where C_7 is a positive constant independent of f.

II. Let G be an open bounded domain in the x,y-plane, whose boundary Γ consists of a finite number of arcs Γ_i, $i = 1,\dots,n$, of class C^{l+1}, with neighboring arcs Γ_i of the boundary forming, at the junction points, angles different from 0. Let the boundary Γ be given by the equations $x = x(s)$, $y = y(s)$, where s is arc length, measured from some point or points if the domain G is multiply connected.

Let now $v(x,y)$ be some infinitely continuously differentiable function in $\overline{G}$. Denote by $\mathbf{n} = \mathbf{n}(s)$ the inward normal to the boundary Γ of the domain G (at those points where it exists). Then, for any pair of nonnegative integers j and k, at each point of the curve Γ in a neighborhood of which the curve is sufficiently smooth, the derivative $\partial^{j+k}v/\partial x^j \partial y^k$ can be expressed as some linear combination of the functions

$$v,\ \frac{\partial v}{\partial s},\ \frac{\partial v}{\partial n},\ \dots,\ \frac{\partial^{j+k}v}{\partial s^{j+k}},\ \frac{\partial^{j+k}v}{\partial s^{j+k-1}\partial n},\ \dots,\ \frac{\partial^{j+k}v}{\partial n^{j+k}}$$

with coefficients independent of the function v, and depending only on the curve Γ in a neighborhood of the given point:

$$\frac{\partial^{j+k}v}{\partial x^j \partial y^k} = L_{jk}\left(v, \frac{\partial v}{\partial s}, \dots, \frac{\partial^{j+k}v}{\partial n^{j+k}}; s\right).$$

Theorem 3. Let $u(x,y) \in W_p^{(l)}(G)$, l a natural number, $1 < p < \infty$. Then the traces

$$\left.\frac{\partial^k u}{\partial n^k}\right|_\Gamma = \varphi_k(s), \quad k = 0,1,\dots,l-1,$$

exist and satisfy the following conditions:

$$1) \quad \varphi_k(s) \in W_p^{(l-k-1/p)}(\Gamma_i), \quad k = 0,1,\dots,l-1; \quad i = 1,\dots,n.$$

Further, if

$$\varphi_{jk} = L_{jk}\left(\varphi_0, \frac{d\varphi_0}{ds}, \dots, \frac{d^{j+k}\varphi_0}{ds^{j+k}}, \varphi_1, \dots, \varphi_{j+k}; s\right),$$

then:

2) $\varphi_{jk} \in W_p^{(1)}(\Gamma), \quad 0 \le j+k \le l-2;$

3) $\varphi_{jk} \in W_p^{(1-1/p)}(\Gamma), \quad j+k = l-1.$

Moreover, the inequality

$$\sum_{k=0}^{l-1}\sum_{i=1}^{n}\|\varphi_k\|_{W_p^{(l-k-1/p)}(\Gamma_i)} + \sum_{j+k=l-1}\|\varphi_{jk}\|_{W_p^{(1-1/p)}(\Gamma)} \le C_8\|u\|_{W_p^{(l)}(G)}$$

holds, where C_8 is a positive constant independent of u.

The converse theorem also holds:

Theorem 4. Let l be a natural number, $1 < p < \infty$, and suppose that on the boundary Γ of the domain G there are given functions $\varphi_k(s)$, $k = 0, 1, \dots, l-1$, such that:

1) $\varphi_k \in W_p^{(l-k-1/p)}(\Gamma_i), \quad k = 0, 1, \dots, l-1; \quad i = 1, \dots, n;$

2) $\varphi_{jk} \in W_p^{(1)}(\Gamma), \quad 0 \le j+k \le l-2;$

3) $\varphi_{jk} \in W_p^{(1-1/p)}(\Gamma), \quad j+k = l-1.$

Then there exists a function $u(x,y) \in W_p^{(l)}(G)$ such that

$$\left.\frac{\partial^k u}{\partial n^k}\right|_{\Gamma} = \varphi_k,$$

$k = 0, 1, \dots, l-1$, and, moreover, the inequality

$$\|u\|_{W_p^{(l)}(G)} \le C_9\left\{\sum_{k=0}^{l-1}\sum_{i=1}^{n}\|\varphi_k\|_{W_p^{(l-k-1/p)}(\Gamma_i)} + \sum_{j+k=l-1}\|\varphi_{jk}\|_{W_p^{(1-1/p)}(\Gamma)}\right\}, \qquad (1)$$

holds, where C_9 is a positive constant independent of the functions φ_k.

Remark 1. If $l = 1$, then conditions 1) and 2) in Theorems 3 and 4 naturally drop out, and condition 3) takes the form

$$u|_{\Gamma} = \varphi_0(s) \in W_p^{(1-1/p)}(\Gamma).$$

Remark 2. If $p \ne 2$, then inequality (1) can be replaced by the inequality

$$\|u\|_{W_p^{(l)}(G)} \le C_{10}\sum_{k=0}^{l-1}\sum_{i=1}^{n}\|\varphi_k\|_{W_p^{(l-k-1/p)}(\Gamma_i)}.$$

This follows from the corollary of Theorems 1 and 2.

For smooth boundaries the assertions of Theorem 4 are known ([4,5,7]); therefore, by the known method of gluing local extensions ([10]), the theorem will be proved if the required function $u(x, y)$ is constructed in a neighborhood of an angular point.

Suppose, for example, that two pieces Γ_1 and Γ_2 adjoin this point. A function $u_1(x, y)$ is constructed, belonging to the class $W_p^{(l)}$ and assuming on Γ_1, together with its normal derivatives, the prescribed boundary values. Analogously, a function $u_2(x, y)$ is constructed. Then the function $u = \beta_1 u_1 + \beta_2 u_2$ is constructed, where the functions $\beta_1(x, y)$ and $\beta_2(x, y)$ are chosen so that the function $u(x, y)$ gives the required extension in a neighborhood of the angular point.

II. Setting $p = 2$ in Theorems 3 and 4, in view of known results we obtain necessary and sufficient conditions for the solution of the first boundary-value problem for the polyharmonic equation $\Delta^m u = 0$ in the class of functions with finite integral

$$\iint_G \sum_{i+j=m} \frac{m!}{i!\,j!} \left(\frac{\partial^m u}{\partial x^i \partial y^j} \right)^2 dx\,dy.$$

IV. Analogous theorems have also been proved by the author for the so-called classes $W_p^{(l)}(G)$ with “degeneration”([3,4,7,11,12]) in the case where the degeneration on approaching the boundary is of power type, and the exponent α depends on the boundary point and $0 \leq \alpha < p-1$. Precise formulations, established for the class $W_p^{(1)}(G)$ with “degeneration,” may be found in the author’ s paper ([12]).

V. A. Steklov Mathematical Institute
Academy of Sciences of the USSR

Received
18 IV 1961

REFERENCES

1. S. L. Sobolev, Some applications of functional analysis in mathematical physics, L., 1950.

2. L. N. Slobodetskii, DAN, 120, No. 3 (1958).

3. A. A. Vasharin, Izv. AN SSSR, 23, 421 (1959).

4. S. V. Uspenskii, DAN, 130, No. 5 (1960).

5. O. V. Besov, DAN, 126, No. 6 (1959).

6. E. Gagliardo, Rend. Semin. Mat. Padova, 27, 284 (1957).

7. P. I. Lizorkin, DAN, 132, No. 3 (1960).

8. S. M. Nikol' skii, Matem. sborn., 40 (82), No. 3, 303 (1956).

9. S. M. Nikol' skii, Matem. sborn., 45 (87), No. 2, 181 (1958).

10. S. M. Nikol' skii, Matem. sborn., 33 (75), 261 (1953).

11. L. D. Kudryavtsev, Tr. Matem. inst. im. V. A. Steklova, 55 (1959).

12. G. N. Yakovlev, Tr. Matem. inst. im. V. A. Steklova, 60 (1961).

Note: Figure translations are in progress. See original paper for figures.