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M. F. POLUYANOVA

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Abstract

Full Text

MATHEMATICS

M. F. POLUYANOVA

ON THE SUMMATION OF THE PRODUCT OF TWO NUMERICAL SERIES

(Presented by Academician A. N. Kolmogorov on 19 VII 1961)

1. In the present note we set forth some theorems on conditions for summability, and also for absolute summability, of the Cauchy product of two numerical series, each of which is summable or absolutely summable by some complex normal triangular method*. The results can be transferred to the case of multiplying any finite number of series. We also consider the relations between the limits of indeterminacy of Voronoi means for the two series and for the product.

By $u_n \times v_n$ everywhere below one should understand $\sum_{k=0}^n u_{n-k} \cdot v_k$. The series $\sum u_n^{**}$ is called absolutely summable by the method $P = (p_{n,k})$ to S (we shall denote

$$\sum u_n = S |p_{n,k}|$$

), if there exists $\lim_{n \rightarrow \infty} U_n^{(p)} = S$ and

$$\sum |U_n^{(p)} - U_{n-1}^{(p)}| < \infty,$$

where

$$U_{-1}^{(p)} = 0, \quad U_n^{(p)} = \sum_{k=0}^n p_{n,k} U_k, \quad U_k = \sum_{i=0}^k u_i.$$

Ordinary summability of $\sum u_n$ to S by the method $(p_{n,k})$ will be denoted by

$$\sum u_n = S(p_{n,k}).$$

By

$$\sum u_n = O(1)(p_{n,k})$$

we denote

$$\sum_{k=0}^n p_{n,k} U_k = O(1).$$

The Voronoi method (W, p) is the method determined by the matrix

$$(p_{n-k} \cdot P_n^{-1}),$$

where

$$P_n = p_0 + p_1 + \dots + p_n \neq 0.$$

The Cesàro method (C, r) is the Voronoi method for

$$p_n = (r+1)(r+2)\dots(r+n)(n!)^{-1}.$$

In the present note, by $k_n, \bar{k}_n, k_{m,n}$ and $\bar{k}_{m,n}$ one should understand the numbers defined by the relations:

$$\sum_{i=r}^n p_{n,i} k_{i,r} = \begin{cases} 1 & \text{for } r = n, \\ 0 & \text{for } 0 \leq r < n; \end{cases} \quad \sum_{i=r}^n q_{n,i} \bar{k}_{i,r} = \begin{cases} 1 & \text{for } r = n, \\ 0 & \text{for } 0 \leq r < n; \end{cases}$$

$$\sum_{i=0}^n p_{n-i} k_i = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \neq 0; \end{cases} \quad \sum_{i=0}^n q_{n-i} \bar{k}_i = \begin{cases} 1 & \text{for } n = 0, \\ 0 & \text{for } n \neq 0, \end{cases}$$

if $p_{n,i}, q_{n,i}, p_k, q_k$ are given.

All quantities that we consider in the present note are assumed to be finite.

* A method of summation of series determined by a matrix $(p_{n,k})$ is called normal triangular if $p_{n,n} \neq 0$ and $p_{n,k} = 0$ for $k > n$ ($n \geq 0$).

** By $\sum u_n$ everywhere below one should understand

$$\sum_{n=0}^{\infty} u_n.$$

Let us fix the methods $(p_{n,k})$ and $(q_{n,k})$ and introduce the following notation:

$R^{(1)}$ is the class of methods $(r_{n,k})$ such that, for any $\sum u_n$ and $\sum v_n$ for which $\sum u_n = U \mid p_{n,k}, \sum v_n = V \mid q_{n,k}$, the relation

$$\sum w_n = \sum u_n \times v_n = U \cdot V \cdot q^{-1}(r_{n,k}),$$

holds, where

$$q = \lim_{n \rightarrow \infty} \sum_{k=0}^n q_{n,k}$$

(it is assumed that q exists and is $\neq 0$).

$R^{(2)}$ is the class of methods $(r_{n,k})$ such that, for any $\sum u_n$ and $\sum v_n$ for which $\sum u_n = U \mid p_{n,k}$ and $\sum v_n = O(1) \mid q_{n,k}$, the relation

$$\sum w_n = \sum u_n \times v_n = O(1) \mid (r_{n,k})$$

holds.

$R^{(3)}$ is the class of methods $(r_{n,k})$ such that, for any $\sum u_n$ and $\sum v_n$ for which $\sum u_n = U \mid p_{n,k}$, $\sum v_n = V \mid q_{n,k}$, the relation

$$\sum w_n = \sum u_n \times v_n = C \mid r_{n,k},$$

holds, where C is some number.

$R^{(4)}$ is the class of methods $(r_{n,k})$ such that, for any $\sum u_n$ and $\sum v_n$ for which $\sum u_n = U(p_{n,k})$, $\sum v_n = V(q_{n,k})$, the relation

$$\sum w_n = \sum u_n \times v_n = U \cdot V(r_{n,k})$$

holds.

The question of summing the product of two series by Voronoi methods was considered by Mears ^(1,2), Borger ⁽³⁾, and Cesàro ⁽⁴⁾.

2. Theorem 1. In order that the method $(r_{n,k})$ belong to the class $R^{(1)}$, it is necessary and sufficient that the following conditions be fulfilled:

$$\sum_{j=0}^n |a_{i,j}^{(n)}| \leq M \quad (\text{for all } n \text{ and all } i); \quad \lim_{n \rightarrow \infty} a_{i,j}^{(n)} = 0 \quad (i, j \text{ fixed}); \quad (1)$$

$$\left| \sum_{l=k}^n c_{n,l} \right| \leq K \quad (\text{for all } k \text{ and } n);$$

$$\lim_{n \rightarrow \infty} c_{n,l} = 0 \quad (\text{for each } l); \quad \lim_{n \rightarrow \infty} \sum_{k=0}^n c_{n,k} = 1, \quad (2)$$

where

$$a_{i,j}^{(n)} = \sum_{k=0}^n r_{n,k} \cdot \sum_{\chi=0}^k \bar{k}_{k-\chi,j} \sum_{r=i}^{\chi} (k_{\chi,r} - k_{\chi-1,r}), \quad (c_{n,l}) = c = R \cdot P^{-1}. \quad (3)$$

In the special case when $(p_{n,k}) = (W, p)$, $(q_{n,k}) = (W, q)$, $(r_{n,k}) = (W, r)$, the following assertions are valid:

Corollary 1,1. In order that the method (W, r) belong to the class $R^{(1)}$, the necessary and sufficient conditions are (1), (2), where now

$$a_{i,j}^{(n)} = \sum_{k=0}^n r_{n-k} \cdot R_n^{-1} \cdot \sum_{\chi=0}^k \bar{k}_{k-\chi,j} \cdot Q_j \sum_{r=i}^{\chi} P_r (k_{\chi-r} - k_{\chi-1-r}).$$

Corollary 1, 2. In order that the method (W, r) , with $r_n = p_n \times q_n$, belong to the class $R^{(1)}$, the necessary and sufficient conditions are:

$$P_{n-k} = o(R_n); \quad q_{n-k} = o(R_n) \quad (\text{for each } k);$$

$$(|p_n| \times |Q_n|) \cdot |R_n^{-1}| \leq L \quad (\text{for all } n);$$

$$\left| \sum_{l=k}^n P_l \cdot q_{n-l} \right| \cdot |R_n^{-1}| \leq L \quad (\text{for all } n \text{ and } k).$$

In the particular case when $p_n \geq 0$, $q_n \geq 0$, $p_n = o(P_n)$, $q_n = o(Q_n)$, we obtain Mears' theorem (1).

Theorem 2. In order that the method $(r_{n,k})$ belong to the class $R^{(2)}$, it is necessary and sufficient that

$$\sum_{j=0}^n |a_{i,j}^{(n)}| \leq M \quad (\text{for all } n \text{ and all } i),$$

where $a_{i,j}^{(n)}$ is defined by (3).

Theorem 3. In order that the method $(r_{n,k})$ belong to the class $R^{(3)}$, it is necessary and sufficient that

$$\sum_{n=0}^m |b_{i,j}^{(n)}| \leq H \quad (\text{for all } m \text{ and all } i, j), \quad (4)$$

where

$$b_{i,j}^{(n)} = \sum_{k=0}^n (r_{n,k} - r_{n-1,k}) \sum_{x=0}^{k-j} \sum_{r=0}^x \bar{k}_{x+j,r+j} \sum_{t=0}^{k-x-j-i} k_{k-x-j,t+i} - k_{k-x-j-1,t+i}.$$

In the particular case when $(p_{n,k}) = (W, p)$, $(q_{n,k}) = (W, q)$, $(r_{n,k}) = (W, r)$, we have

Corollary 3.1. In order that the method (W, r) belong to the class $R^{(3)}$, it is necessary and sufficient that condition (4) be fulfilled for

$$r_{n,k} = r_{n-k} \cdot (r_n \times 1)^{-1}, \quad k_{n,r} = k_{n-r} \cdot P_r, \quad \bar{k}_{n,r} = \bar{k}_{n-r} \cdot Q_r.$$

In the particular case when $r_n = p_n \times q_n$ and (W, r) includes absolutely both (W, p) and (W, q) , we obtain Mears' theorem (2).

Theorem 4. In order that the method $(r_{n,k})$ belong to the class $R^{(4)}$, the conditions

$$\lim_{n \rightarrow \infty} \sum_{i=0}^n \sum_{j=0}^n d_{i,j}^{(n)} = 1, \quad \sum_{i=0}^n \sum_{j=0}^n |d_{i,j}^{(n)}| \leq M \quad (\text{for all } n); \quad \lim_{n \rightarrow \infty} d_{i,j}^{(n)} = 0 \quad (5)$$

$(i, j \text{ fixed});$

$$\lim_{n \rightarrow \infty} \sum_{i=0}^n |d_{i,j}^{(n)}| = 0 \quad (\text{for all } j); \quad \lim_{n \rightarrow \infty} \sum_{j=0}^n |d_{i,j}^{(n)}| = 0 \quad (\text{for all } i), \quad (6)$$

where

$$d_{i,j}^{(n)} = \sum_{k=0}^n r_{n,k+i+j} \sum_{x=0}^k (k_{k-x+i,i} - k_{k-1-x+i,i}) \bar{k}_{x+j,j}.$$

are sufficient.

If, in addition, $d_{i,j}^{(n)} \geq 0$, then these conditions are also necessary.

In the particular case when $(p_{n,k}) = (W, p)$, $(q_{n,k}) = (W, q)$, $(r_{n,k}) = (W, r)$, we have

Corollary 4.1. In order that the method (W, r) belong to the class $R^{(4)}$ for

$$r_{n,k} = r_{n-k} \cdot (r_n \times 1)^{-1}, \quad k_{n,r} = k_{n-r} \cdot P_r, \quad \bar{k}_{n,r} = \bar{k}_{n-r} \cdot Q_r, \quad (7)$$

it is sufficient that conditions (5), (6), (7) be fulfilled.

Moreover, for $r_n = p_n \times Q_n$ the conditions (5) and (6) here are also necessary. Consequently, the following assertion is valid:

Corollary 4.2. In order that the method (W, r) , with $r_n = p_n \times Q_n$, belong to the class $R^{(4)}$, it is necessary and sufficient that the following conditions hold:

$$P_{n-k} = o(R_n), \quad Q_{n-k} = o(R_n) \quad (\text{for each } k), \\ (|P_n| \times |Q_n|) \cdot |R_n^{-1}| \leq M \quad (\text{for all } n).$$

In the special case when $p_n \geq 0$, $q_n \geq 0$, $p_n = o(P_n)$, $q_n = o(Q_n)$, we obtain Mears' theorem ⁽¹⁾. In another special case, namely for the Cesàro methods (C, r) and (C, s) with $\text{Re } r > -1$ and $\text{Re } s > -1$, we obtain Borger' s theorem ⁽³⁾, p. 52).

3. Theorem 5. Let u_n, v_n, p_n, q_n ($n \geq 0$) be real, $P_n > 0$, $Q_n > 0$, $r_n = p_n \times Q_n$,

$$P_{n-k} = o(R_n), \quad Q_{n-k} = o(R_n) \quad (\text{for each } k); \\ U_n^{(p)} = (p_n \times u_n \times 1) \cdot P_n^{-1}, \quad V_n^{(q)} = (q_n \times v_n \times 1) \cdot Q_n^{-1}, \\ W_n^{(R)} = (r_n \times u_n \times v_n \times 1) \cdot (r_n \times 1)^{-1},$$

$$\overline{\lim}_{n \rightarrow \infty} U_n^{(p)} = \bar{\alpha}, \quad \underline{\lim}_{n \rightarrow \infty} U_n^{(p)} = \underline{\alpha}, \quad \overline{\lim}_{n \rightarrow \infty} V_n^{(q)} = \bar{\beta}, \quad \underline{\lim}_{n \rightarrow \infty} V_n^{(q)} = \underline{\beta}.$$

Then the inequality

$$\min\{\bar{\alpha}\bar{\beta}, \bar{\alpha}\underline{\beta}, \underline{\alpha}\bar{\beta}, \underline{\alpha}\underline{\beta}\} \leq \underline{\lim} W_n^{(R)} \leq \overline{\lim} W_n^{(R)} \leq \max\{\bar{\alpha}\bar{\beta}, \bar{\alpha}\underline{\beta}, \underline{\alpha}\bar{\beta}, \underline{\alpha}\underline{\beta}\}. \quad (8)$$

In the special case of the Cesàro methods (C, r) and (C, s) , inequality (8) is valid for $r > -1$, $s > -1$. In another special case, when $p_n \geq 0$, $q_n \geq 0$ ($n \geq 0$), $p_n = o(P_n)$, $q_n = o(Q_n)$, we obtain Mears' theorem ⁽¹⁾.

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Note: Figure translations are in progress. See original paper for figures.

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