B. P. DEMIDOVICH

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Abstract

Full Text

B. P. DEMIDOVICH

ON BOUNDED SOLUTIONS OF A CERTAIN QUASI-LINEAR SYSTEM

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1°. In this paper we generalize the theorem of Farnell, Langhenhop, and Levinson ([1]) on the existence and stability of forced harmonic oscillations of weakly nonlinear systems of differential equations (see Theorems 1, 2, and 3).

2°. Consider the linear system

$$\frac{dx}{dt} = P(t)x, \tag{1}$$

where x is an n-dimensional vector (an $n \times 1$ matrix) and $P(t) \in C(-\infty, +\infty)$ is an $n \times n$ matrix.

Definition. We shall say that system (1) (or the matrix $P(t)$) has the property (α, β) if its Cauchy matrix $X(t, t_0) = X(t)X^{-1}(t_0)$ decomposes into two subsystems:

$$X_\alpha(t, t_0) = X(t)AX^{-1}(t_0), \qquad X_\beta(t, t_0) = X(t)BX^{-1}(t_0)$$

(A and B are constant $n \times n$ matrices, with $A + B = I$) such that

$$\|X_\alpha(t, t_0)\| \leq ae^{-\alpha(t-t_0)} \qquad \text{for } t \geq t_0; \tag{2'}$$

$$\|X_\beta(t, t_0)\| \leq be^{\beta(t-t_0)} \qquad \text{for } t \leq t_0. \tag{2''}$$

Here a, b, α, β are positive constants.

Conditions $(2')$ and $(2'')$ are a generalization of the known conditions of K. P. Persidskii ([2]) and are satisfied, for example, for a system reducible to a constant matrix whose characteristic numbers have nonzero real parts.

Lemma 1. Suppose that the matrix $P(t)$ has the property (α, β). Then there exists an $n \times n$ matrix $G(t, t_1) \in C^1$ for $t \neq t_1$ $(-\infty < t < +\infty,\ -\infty < t_1 < +\infty)$ such that:

1) $G(t, t-0) - G(t, t+0) = I$;
2) $G'_t(t, t_1) = P(t)G(t, t_1)$ for $t \neq t_1$,

$$G'_{t_1}(t,t_1) = -G(t,t_1)P(t_1) \quad \text{for } t \neq t_1;$$

3) $\|G(t,t_1)\| \leq ce^{-\gamma|t-t_1|}$ (c and γ are positive constants);

4) any nonhomogeneous system

$$\frac{dy}{dt} = P(t)y + f(t), \tag{3}$$

where $f(t) \in C(-\infty,+\infty)$, has a solution bounded on $(-\infty,+\infty)$,

$$\hat{y}(t) = \int_{-\infty}^{+\infty} G(t,t_1)f(t_1)\,dt_1. \tag{4}$$

under the condition that the integral (4) converges uniformly in t on every finite interval and is bounded on $(-\infty,+\infty)$.

For the proof we set

$$G(t,t_1) = \begin{cases} X_\alpha(t,t_1), & \text{for } t > t_1, \\ -X_\beta(t,t_1), & \text{for } t < t_1. \end{cases}$$

Lemma 2. If $P(t)$ and $f(t)$ are almost periodic and system (1) has no nontrivial bounded solutions, then the solution $\hat{y}(t)$ (4) is also almost periodic.

The lemma is a modified Favard theorem ([3]).

3°. Consider the quasilinear system

$$\frac{dy}{dt} = P(t)y + f(\omega t, y) + e(\omega t), \tag{5}$$

where $P(t) \in C(-\infty,+\infty)$, is bounded* and has the property (α,β); ω is a large parameter $(\omega \geq \omega_0 > 0)$; $f(t,y) \in C(|t| < +\infty, \|y\| < +\infty)$, satisfies a Lipschitz condition in y with constant L, and

$$f(t,0) \equiv 0,$$

and $e(t)$ is a continuous function with bounded integral

$$E(t) = \int_0^t e(t_1)\,dt_1.$$

Theorem 1. If the constant L is sufficiently small, then system (5) has at least one solution $y = \hat{y}(t)$, bounded on $(-\infty,+\infty)$, such that

$$\|\hat{y}(t)\| < \frac{\Gamma}{\omega} \sup_t \|E(t)\|,$$

where Γ is a constant depending only on the matrix $P(t)$.

This result is analogous to Perron' s theorem ([4]), but is not its consequence, since $e(t)$ may be unbounded.

The proof is carried out by considering the integral equation

$$y(t) = \int_{-\infty}^{+\infty} G(t, t_1) \left[f(\omega t_1, y(t_1)) + e(\omega t_1)\right] dt_1,$$

which is solved by the method of successive approximations.

Theorem 2. If the matrix $P(t)$ is almost periodic (periodic with period T/ω), and also $f(t, y)$, $e(t)$, and $E(t)$ are almost periodic in t (periodic with period T), and the homogeneous system (1) with matrix $P(t)$ has no bounded nontrivial solutions, then the bounded solution $\hat{y}(t)$ of system (5) is almost periodic (periodic with period T/ω).

For the case of a constant matrix $P(t) = \text{const}$, the existence of an almost periodic solution for quasilinear systems analogous to (5) was established by G. I. Biryuk ([5]), by the author ([6]), and by Langenhop ([7]).

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[*] One may allow a weak unboundedness of the matrix $P(t)$, namely, it suffices to assume that $P(t) = P_0(t) + P_1(t)$, where $P_0(t)$ is a bounded matrix and $P_1(t)$ is absolutely integrable on $(-\infty, +\infty)$.

Theorem 3. If

$$\operatorname{rank} X_a(t_0, t_0) = n,$$

where n is the order of system (1), then the bounded solution $\hat{y}(t)$ is exponentially stable in the sense of Lyapunov as $t \to +\infty$.

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Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.