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Abstract

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ON STRUCTURAL PROPERTIES OF SEMIGROUPS

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In the present note we consider certain structural properties of semigroups* in connection with the notions of structural characterizability and structural definability (definitions are given below). The basic semigroup-theoretic and lattice-theoretic notions are assumed to be known; the corresponding definitions may be found in books ([1],[2]). We shall also use generally accepted notation.

Elements a and $\bar{a}$ of an arbitrary lattice with zero 0 and unit I are called mutually complementary if $a \wedge \bar{a} = 0$, $a \vee \bar{a} = I$. The principal ideal in a lattice ([2]) generated by an element t will be denoted by (t). By (t_1, t_2) we shall denote the principal ideal generated by the element $t_1 \vee t_2$. An element b of an arbitrary lattice will be called singly covering if the ideal (b) has a greatest element distinct from b**. By a local system in a complete semilattice with respect to sum with unit I we mean a set L of elements of this semilattice such that the sum of all elements of this set is equal to I, and every pair of elements of L is contained in some third element of L (cf. ([3])).

By $\Sigma(\Gamma)$ we shall denote the partially ordered by inclusion set of all subsemigroups of the semigroup Γ. $\Sigma(\Gamma)$ is a complete semilattice with respect to the operation of taking the composite of subsemigroups, but it will not always be a lattice. Therefore it is often expedient to consider the extended set $\Sigma'(\Gamma)$, obtained by adjoining the empty set to $\Sigma(\Gamma)$. $\Sigma'(\Gamma)$ is a complete lattice. $\Sigma(\Gamma)$ and $\Sigma'(\Gamma)$ simultaneously possess or do not possess a number of properties: finiteness, infiniteness, linear ordering, etc.; therefore, in studying the structural properties of a semigroup, it is often immaterial whether we consider $\Sigma(\Gamma)$ or $\Sigma'(\Gamma)$. However, in $\Sigma(\Gamma)$ the operation of intersection is not always defined, which immediately presents an inconvenience in studying such properties where this operation is involved, for example distributivity, Dedekindness. In such questions it is better to deal with $\Sigma'(\Gamma)$.

For one and the same semigroup Γ, $\Sigma(\Gamma)$ and $\Sigma'(\Gamma)$ are almost always nonisomorphic. Specifically, the following is true:

Theorem 1. $\Sigma(\Gamma)$ is isomorphic to $\Sigma'(\Gamma)$ if and only if Γ is a group of type p^∞.

At the same time it is obvious that if Γ_1 and Γ_2 are two semigroups, then $\Sigma(\Gamma_1)$ and $\Sigma(\Gamma_2)$ are isomorphic if and only if $\Sigma'(\Gamma_1)$ and $\Sigma'(\Gamma_2)$ are isomorphic.

* That is, properties connected with the structure of the partially ordered set of subsemigroups of a semigroup.

** In this note we shall consider lattices of subsemigroups; therefore henceforth their elements will be denoted by capital Latin letters.

Semigroups Γ_1 and Γ_2 are called **structurally isomorphic** if $\Sigma(\Gamma_1)$ is isomorphic to $\Sigma(\Gamma_2)$ (by the preceding remark this is equivalent to an isomorphism of $\Sigma(\Gamma_1)$ and $\Sigma'(\Gamma_2)$). A number of elementary properties of structural isomorphisms are considered in ([4,6,1]).

Let $\mathfrak{A}$ and $\mathfrak{B}$ be two classes of semigroups (closed under isomorphisms), with $\mathfrak{A}$ a subclass of the class $\mathfrak{B}$. We shall say that the class $\mathfrak{A}$ is **structurally characterizable in the class** $\mathfrak{B}$ if one can specify conditions, formulated in terms of the theory of structures, such that for every semigroup Γ from the class $\mathfrak{B}$, $\Sigma'(\Gamma)$ satisfies these conditions if and only if Γ belongs to the class $\mathfrak{A}$. The class $\mathfrak{A}$ is **structurally determined in the class** $\mathfrak{B}$ if every semigroup from the class $\mathfrak{B}$ that is structurally isomorphic to some semigroup from the class $\mathfrak{A}$ itself belongs to the class $\mathfrak{A}$. It is clear that if the class $\mathfrak{A}$ is structurally characterizable in the class $\mathfrak{B}$, then it is also structurally determined in this class.

Finding structural characteristics for structurally determined classes must be regarded as a quite natural and important problem.

Especially important is the case when $\mathfrak{B}$ is the class of all semigroups. In this case we shall simply speak of the **structural characterizability** and **structural determinability** of the class $\mathfrak{A}$ (concerning the latter notion see ([1]), where somewhat different terminology is used). Many classes of semigroups are not structurally determined. In such cases it is expedient to find sufficiently broad and natural superclasses of them in which structural determinability already holds. We shall give some examples of this kind below. When the class consists of a single semigroup Γ, it is, of course, necessary to speak of the structural characterizability or structural determinability of the semigroup Γ.

All structurally determined classes listed in ([1]) (Chap. III, Sec. 7.9) in fact admit a simple structural characteristic. We shall note only a few examples.

Lemma 1.
a) A semigroup Γ is a periodic semigroup if and only if $\Sigma'(\Gamma)$ has atoms and every principal ideal in $\Sigma'(\Gamma)$ contains at least one atom.

b) In particular, Γ is a periodic semigroup with one idempotent if and only if the structure $\Sigma'(\Gamma)$ has a unique atom contained in each of its principal ideals. For this it is also necessary and sufficient that $\Sigma(\Gamma)$ have a zero.

c) All elements of the semigroup Γ have infinite order if and only if there are no atoms in $\Sigma'(\Gamma)$.

Using the proofs of the structural determinability of certain classes of semigroups considered in ([5]) (see also ([1])), one can establish the corresponding structural characteristics of these classes (Lemma 2, Theorems 2 and 3).

Lemma 2.
In order that a semigroup Γ be an infinite cyclic group, it is necessary and sufficient that $\Sigma'(\Gamma)$ satisfy the following conditions:

a) $\Sigma'(\Gamma)$ contains a unique atom E.

b) In $\Sigma'(\Gamma)$ there are two mutually complementary one-covering elements A and $\bar{A}$, generating infinite principal ideals.

c) Every one-covering element of $\Sigma'(\Gamma)$ distinct from E belongs to one of the ideals (A), $(\bar{A})$.

d) If A' is an element covered by A, then $A \in (A', \bar{A})$.

Theorem 2.
In order that a semigroup Γ be a torsion-free group, it is necessary and sufficient that $\Sigma'(\Gamma)$ contain a unique atom E, and that every one-covering element of $\Sigma'(\Gamma)$ can be included in a substructure satisfying conditions a)-d) of Lemma 2.

Theorem 3. *In order that a semigroup Γ be a nonperiodic group, it is necessary and sufficient that $\Sigma'(\Gamma)$ satisfy the following conditions:*

a) $\Sigma'(\Gamma)$ contains a unique atom E.

b) $\Sigma'(\Gamma)$ contains single-covering elements that generate infinite principal ideals, and each such element can be included in a substructure satisfying conditions a)—d) of Lemma 2.

c) Every single-covering element $C \in \Sigma'(\Gamma)$ that generates a finite principal ideal is contained in some ideal (A, B), where $C \bar{\in} (A)$, and B is a single-covering element generating an infinite principal ideal.

Although periodic semigroups with one idempotent are characterized structurally (Lemma 1, b)), their most important special cases—periodic groups and nilsemigroups (see ([7,8]))—are not structurally determined, as simple examples show. However, one can point out sufficiently large and natural superclasses in which even a structural characterization of these classes takes place. For periodic groups such a superclass will be the class of semigroups with identity.

Theorem 4. *In order that a semigroup Γ with identity be a periodic group, it is necessary and sufficient that $\Sigma(\Gamma)$ have a zero.*

Some important subclasses of the class of periodic groups are also structurally characterized in the class of semigroups with identity.

Theorem 5. *In order that a semigroup Γ with identity be a periodic locally cyclic group, it is necessary and sufficient that $\Sigma(\Gamma)$ be a distributive structure with zero.*

Theorem 6. *In order that a semigroup* Γ *with identity be a finite cyclic group, it is necessary and sufficient that* $\Sigma(\Gamma)$ *be a finite distributive structure.*

For nilsemigroups such a natural superclass will be the class of semigroups with zero.

Theorem 7. *In order that a semigroup* Γ *with zero be a nilsemigroup, it is necessary and sufficient that* $\Sigma(\Gamma)$ *have zero.*

In the proof of these theorems an essential role is played by the following

Lemma 3. *A semigroup* Γ *will be a periodic semigroup with one idempotent if and only if it is an extension (see ([9])) of a periodic group* G *by means of a nilsemigroup. For a given semigroup* Γ*, the group* G *is determined uniquely and is the smallest ideal in* Γ *(the kernel of* Γ*).*

Nilpotent and locally nilpotent semigroups (see ([7,8])), which constitute important subclasses of the class of nilsemigroups, are also structurally characterized in the class of semigroups with zero.

Theorem 8. *In order that a semigroup* Γ *with zero be a nilpotent semigroup, it is necessary and sufficient that* $\Sigma(\Gamma)$ *have a zero* 0 *and possess a finite chain of elements*

$$0 = H_0 < H_1 < \ldots < H_k = I$$

(I *is the identity of* $\Sigma(\Gamma)$*), all intervals* $[H_i, H_{i+1}]$ *of which are structures with complements,* $i = 0, 1, \ldots, k-1$*.*

Theorem 9. *In order that a semigroup* Γ *with zero be a locally nilpotent semigroup, it is necessary and sufficient that in* $\Sigma(\Gamma)$ *there exist a local system all of whose elements generate principal ideals that are structures satisfying the conditions of Theorem 8.*

We say that an element x of an arbitrary semigroup Γ is its **divisor** if there exist elements $g_1, g_2 \in \Gamma$ such that at least one of the equalities $x = g_1 x$, $x = x g_2$, $x = g_1 x g_2$ holds.

In the proof of Theorem 8 an essential role is played by the following lemma, which is also of independent interest.

Lemma 4. *Let* H *be a subsemigroup of an arbitrary semigroup* Γ *such that the set* $\Gamma \setminus H$ *contains no elements that are their own divisors. Then the interval* $[H, \Gamma]$ *will be a structure with complements if and only if* $\Gamma^2 \subseteq H$*.*

Let Ω be an arbitrary nonempty set. By V_Ω denote the semigroup of all ordered pairs $[\alpha, \beta]$ $(\alpha, \beta \in \Omega)$, with an adjoined symbol 0, under multiplication:

$$[\alpha,\beta]\cdot[\gamma,\delta] = \begin{cases} [\alpha,\delta], & \text{if } \beta = \gamma, \\ 0, & \text{if } \beta \neq \gamma, \end{cases} \qquad [\alpha,\beta]\cdot 0 = 0 \cdot [\alpha,\beta] = 0 \cdot 0 = 0.$$

E. S. Lyapin in ([10]) studied the properties of the semigroup V_Ω, obtained an abstract characterization, and elucidated the important role of V_Ω as a subsemigroup in the semigroup of all one-to-one partial transformations of the set Ω.

Theorem 10. *Let the set Ω consist of more than one element. Then the semigroup V_Ω is structurally determined.*

The very long proof of this theorem is based on the special case, considered at the beginning, in which Ω consists of two elements.

Using the method of the work ([4]) (see also ([1])), one can establish the structural determinacy of the class of free semigroups considered by us in ([11]):

Theorem 11. *Every mixed free semigroup is structurally determined.*

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Note: Figure translations are in progress. See original paper for figures.