Mathematics

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Abstract

Full Text

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ON CERTAIN ESTIMATES OF A DIFFERENTIABLE FUNCTION

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1. In the present note two estimates* are given for a real differentiable function satisfying certain conditions, and an application of one of them to a multipoint boundary-value problem is indicated.

Theorem 1. Let $x(t)$ be continuously differentiable n times on the interval $[a, b]$ and satisfy the conditions

$$x(a_1) = x'(a_2) = \dots = x^{(n-1)}(a_n) = 0, \tag{1}$$

where $a_1, a_2, \dots, a_n$ are certain points of the interval $[a, b]$. Then on $[a, b]$ the estimate

$$|x(t)| \le C_n (b-a)^n \max_{a \le t \le b} |x^{(n)}(t)|, \qquad a \le t \le b, \tag{2}$$

holds, where the numbers $C_1, C_2, \dots$ are determined from the expansion

$$\operatorname{tg} t + \sec t = 1 + \sum_{k=1}^{\infty} C_k t^k \quad \left(|t| < \frac{\pi}{2}\right). \tag{3}$$

In accordance with (3), the values C_k can be expressed in terms of the Bernoulli numbers B_k ($B_1 = 1/6$, $B_2 = 1/30, \dots$) and the Euler numbers E_k ($E_1 = 1$, $E_2 = 5, \dots$) as follows ($n = 1, 2, \dots$):

$$C_{2n-1} = \frac{2^{2n}(2^{2n}-1)B_n}{(2n)!}, \qquad C_{2n} = \frac{E_n}{(2n)!}.$$

Theorem 2. Let $x(t)$ satisfy the conditions (1), and suppose that one of the inequalities

$$(a \le)\ a_2 \le a_3 \le \dots \le a_n\ (\le b),$$

$$(a \le)\ a_n \le a_{n-1} \le \dots \le a_2\ (\le b).$$

is fulfilled. Then on $[a,b]$ the estimate

$$|x(t)| \le \frac{1}{n\left[\frac{n-1}{2}\right]!\left[\frac{n}{2}\right]!}(b-a)^n \max_{a\le t\le b}|x^{(n)}(t)|, \qquad a \le t \le b. \tag{4}$$

holds.

The values of the constants in the estimates (2), (4) cannot be improved.

2. Let the multipoint boundary-value problem be considered:

$$x^{(n)} + p_1(t)x^{(n-1)} + \dots + p_n(t)x = f(t), \tag{5}$$

$$x(a_1) = A_1, \quad x(a_2) = A_2, \dots, \quad x(a_n) = A_n, \tag{6}$$

$$a \le a_1 < a_2 < \dots < a_n \le b,$$

where $p_1(t), \dots, p_n(t), f(t)$ are continuous on $[a,b]$. One may also not exclude the case of coincidence of several a_i; in that case, at one point values are prescribed both for $x(t)$ and for its successive derivatives (in accordance with the multiplicity of a_i).

* The first of them, as has become clear, was obtained by S. N. Bernstein as early as 1910.

Set

$$h = a_n - a_1, \qquad \max_{a_1\le t\le a_n}|p_i(t)| = P_i \quad (i = 1, 2, \dots, n).$$

Vallée-Poussin (^{1}) (see also (^{2})) indicated the following sufficient condition for the solvability of problem (5)–(6):

$$\sum_{k=1}^{n}\frac{1}{k!}P_k h^k \leqslant 1. \tag{7}$$

The author (^{3}) (and, independently of him, V. G. Maz'ya) found a less restrictive condition than (7) for the solvability of problem (5)–(6):

$$\sum_{k=1}^{n-1}\frac{1}{k!}P_k h^k + \frac{(n-1)^{n-1}}{n^n n!}P_n h^n \leqslant 1. \tag{8}$$

With the aid of Theorem 2 the following proposition can be proved.

Theorem 3. In order that problem (5)–(6) be solvable for arbitrary $A_1, A_2, \dots, A_n$ and arbitrary continuous $f(t)$, it is sufficient that the inequality

$$\sum_{k=1}^{n} \frac{1}{2^k k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} P_k h^k \leqslant 1 \tag{9}$$

hold.

From the relation

$$\frac{1}{2^k k \left[\frac{k-1}{2}\right]! \left[\frac{k}{2}\right]!} = \frac{C_{k-1}^{\left[\frac{k}{2}\right]}}{2^k} \frac{1}{k!} \leqslant \frac{1}{2} \frac{1}{k!}$$

it follows that in inequality (9) the coefficients of all powers of h are smaller than the corresponding coefficients in inequality (7).

It is easy to verify that conditions (8) and (9) are (for $n > 2$) mutually independent.

3. With the aid of the same Theorem 2 one can obtain a strengthening of Vallée-Poussin’ s uniqueness theorem (see (2)) for a nonlinear boundary-value problem. Namely, the following proposition is valid:

Theorem 4. Let the right-hand side of the equation

$$x^{(n)} = f(t, x, x', \dots, x^{(n-1)}) \tag{10}$$

satisfy the condition

$$|f(t, v_0, v_1, \dots, v_{n-1}) - f(t, u_0, u_1, \dots, u_{n-1})| \leqslant \sum_{k=0}^{n-1} P_{n-k} |v_k - u_k|$$

and, in addition, suppose that one of inequalities (8), (9) is fulfilled. Then equation (10) has no more than one solution satisfying conditions (6).

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References

[1] Ch. J. De la Vallée Poussin, *J. de Math. pures et appl.* (9), 8 (1929).
[2] J. Sansone, *Ordinary Differential Equations*, IL, 1953.
[3] A. Yu. Levin, *Scientific Reports of the Higher School. Physics and Mathematics*, No. 5 (1958).
[4] S. N. Bernstein, *Collected Works*, 2, article 100, 1954, p. 497.

Note: Figure translations are in progress. See original paper for figures.

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