Mathematics

1961

SovietRxiv

Abstract

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Mathematics

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On the Dirichlet and Neumann Problems on Triangular and Hexagonal Grids

(Presented by Academician S. L. Sobolev on 12 XII 1960)

It is known (^{2}) that on a hexagonal grid the local approximation of the Laplace operator by differences is performed using 4 points, but the error of this approximation is $O(h)$, while on a triangular grid, although the error of the approximation of the Laplace operator by differences is of order $O(h^4)$, the approximation is performed using 7 points. In the present note it is shown that the error between the approximate and sufficiently smooth exact solution arising in the boundary-value problems under consideration when the Laplace operator is replaced by a difference operator defined on a hexagonal grid is of order h^2; this will follow from the fact that the residual in the approximation of the Laplace operator by differences is represented in divergent form and has opposite sign at neighboring points, up to terms $O(h^2)$. It is also shown that from systems of equations whose solutions approximate, on a triangular grid, the solutions of the problems under consideration, one can pass to another system (for the h-conjugate function defined on a hexagonal grid), each equation of which contains only 4 unknowns.

Let U be a harmonic function, and V its conjugate, defined inside a simply connected domain Ω in the plane (x_1, x_2). Construct in the plane (x_1, x_2) a triangular grid D_h with mesh size $h = h_1$. Denote by Ω_h the set of centers of the triangles; it forms a hexagonal grid with mesh size $h_2 = h_1/\sqrt{3}$. Denote by B_h the set of midpoints of the segments forming the grids D_h, Ω_h. Consider a quadruple of points of the sets Ω_h, D_h lying in Ω: let the points $A, B \in \Omega_h$ be two neighboring points, the points $C, D \in D_h$ be at distance $h_1/\sqrt{3}$ from the points A, B, and let $\mathbf{n}, \mathbf{s}$ be unit vectors in the directions, respectively, $\overline{BA}$, $\overline{DC}$, forming a right-handed coordinate system (n, s). Then the equalities

$$\frac{\partial U}{\partial n} = \frac{\partial V}{\partial s}, \qquad \frac{\partial U}{\partial s} = -\frac{\partial V}{\partial n} \tag{1}$$

will be replaced inside Ω by the approximate ones

$$h_2^{-1}(u_A - u_B) = h_1^{-1}(v_C - v_D),$$

$$h_1^{-1}(u_C - u_D) = h_2^{-1}(v_B - v_A). \tag{2}$$

The errors $\eta = U - u,\ \zeta = V - v$ thereby admitted will satisfy the system of equations

$$h_2^{-1}(\eta_A - \eta_B) - h_1^{-1}(\zeta_C - \zeta_D) = \psi_1,$$

$$h_1^{-1}(\eta_C - \eta_D) + h_2^{-1}(\zeta_A - \zeta_B) = \psi_2, \tag{3}$$

where, in the local coordinate system,

$$\psi_1(0,0) = \frac{h_2^2}{6}\frac{\partial^3}{\partial n^3}\left[U(\theta_1 h_2, \theta_2 h_1)\right] = -\frac{h_2^2}{6}\frac{\partial^3}{\partial s^3}\left[V(\theta_1 h_2, \theta_2 h_1)\right],$$

$$\psi_2(0,0) = \frac{h_2^2}{6}\frac{\partial^3}{\partial s^3}\left[U(\theta_3 h_2, \theta_4 h_1)\right], \tag{4}$$

$$|\theta_i| \leqslant 1, \quad i = 1,2,3,4.$$

It is enough to consider only two cases for system (2):

1) For a function u, defined on Ω_h, we solve the difference Dirichlet problem; then for the h-conjugate function v, defined on D_h, we obtain the difference Neumann problem. For the domain Ω, from the set D_h we distinguish the interior nodes D_h^1 and the boundary nodes Γ_h^1. The function v is defined on

   $$D_h^2 = D_h^1 + \Gamma_h^1.$$

   From the set Ω_h^2—the centers of triangles all of whose vertices belong to D_h^2—we distinguish S_h^1, the set of centers of those triangles for which two vertices belong to Γ_h^1; let

   $$\Omega_h^1 = \Omega_h^2 - S_h^1.$$

   The function u is defined on Ω_h^2.

2) For a function v, defined on D_h^2, we solve the difference Dirichlet problem; then for the h-conjugate function u we obtain the difference Neumann problem. We regard the function u as defined on the set $\Omega_h^2 + S_h^2$, where $S_h^2 \in \Omega_h$ and is at distance h_2 from Ω_h^2.

All newly introduced sets are assumed to lie in Ω. The values of u on S_h^1 determine on Γ_h^1 the value of $\Delta v/\Delta n$ for v, and the values of $\Delta u/\Delta n$ on S_h^2 determine on Γ_h^1 the values of v up to an additive constant (*). In addition, as is easy to see, the function u satisfies at interior points the difference Laplace equation for the hexagonal grid:

$$\Delta_h' u = 0,$$

while the function v satisfies the difference Laplace equation on the triangular grid:

$$\Delta_h'' v = 0.$$

Thus we find that to each of our boundary-value problems for the function v there corresponds a conjugate boundary-value problem for the function u.

We now estimate the quantity η. To this end we write the expression for $\Delta_h' \eta$ at a point $x \in \Omega_h^1$. Let $\mathbf{n}_i$, $i = 1, 2, 3,$—be the unit vectors issuing from the point x in the directions of the three points of the set Ω_h^2 adjacent to x; then, substituting indices i for the derivatives in expression (4), we obtain

$$\Delta_h' \eta = \frac{4}{3} h_2^{-1} \sum_{i=1}^{3} \psi_1 \left(\frac{1}{2} h_2 \mathbf{n}_i \right) = \psi_3, \qquad \eta\big|_{S_h^1} = \eta_0,$$

let

$$|\eta_0| \leqslant \varepsilon,$$

and if $\eta = \eta_1 + \eta_2$, where

$$\Delta_h' \eta_1 = \psi_3, \qquad \eta_1\big|_{S_h^1} = 0;$$

$$\Delta_h' \eta_2 = 0, \qquad \eta_2\big|_{S_h^1} = \eta_0, \tag{5}$$

then

$$|\eta_2| \leqslant \varepsilon.$$

From equations (5) we obtain that

$$h_2^2 \sum_{\Omega_h^1} \eta_1 \Delta_h \eta_1 = h_2^2 \sum_{\Omega_h'} \eta_1 \psi_3.$$

Summing by parts on both sides of this equality and denoting by η_{1n_i} the divided differences of the function η_1 in the direction $\mathbf{n}_i$, we obtain the estimate

$$h_2^2 \sum_{B_h} \sum_{i=1}^{3} (\eta_{1n_i})^2 \leqslant c_1 h_2^2 \left[h_2^2 \sum_{B_h} \sum_{i=1}^{3} \left(\frac{\partial^3 U}{\partial n_i^3} \right)^2 \right].$$

from which we obtain that the right-hand side of this inequality will be of order $O(h_2^2)$, if the third derivatives of the function U are bounded or have singularities of the form $\rho^{-\lambda_k}$ at a finite number of boundary points s_k, where ρ_k is the distance to s_k, and $0 \leqslant \lambda_k < {}^1/_2$; hence, as in (4), we obtain that in every interior subdomain of the domain Ω

$$\eta = O(h_2^2 + \varepsilon).$$

Let $U \subset H(4, A, {}^1/_2)$ (1), and let the functions η', ξ_1 satisfy system (3); define the function η' on S_h^1 as follows: let $s_0 \in S_h^1$ and $\eta'(s_0) = 0$, and at the remaining points of S_h^1 define η' in the direction of a left traversal of S_h^1 so that $\Delta\xi_1/\Delta n = 0$ on Γ_h^1. This can be done, using equations (3), at all points of Γ_h^1 except one; at this point we additionally require that $\xi_1 = 0$; note that at it $\Delta\xi_1/\Delta n = O(h_1^2)$. Then in Ω_h^2, $\eta_1 - \eta' = O(h_2^2)$, and at any point $x \in D_h^1$

$$\Delta_h'' \xi_1 = \psi_4, \tag{6}$$

where $\psi_4 = O(h^{5/2})$.

Let s_i, $i = 1, 2, 3$, be the unit vectors issuing from the point x in the directions of the segments e_{ik}, $k = 1, 2, \dots, N_i$, $i = 1, 2, 3$, forming the triangular grid of the set D_h^2, and let the angles between s_1, s_2 and s_2, s_3, measured counterclockwise, be equal to $\pi/3$; then the quantity ψ_4, using the fact that

$$\sum_{i=1}^{3} \frac{\partial^4 V}{\partial s_i^4} = 0,$$

can be represented as

$$\psi_4 = \sum_{i=1}^{3} \psi_{5i},$$

where

$$\psi_{5i} = \frac{2}{3h_1} \left(\psi_1 \left(x + \frac{1}{2} h_1 s_i \right) - \psi_1 \left(x - \frac{1}{2} h_1 s_i \right) \right) + \frac{h_1^2}{27} \frac{\partial^4 V(x)}{\partial s_i^4},$$

and therefore, from equation (6), we obtain

$$h_1^2 \sum_{B_h} \sum_{i=1}^{3} (\xi_{1s_i})^2 = -2h_1^2 \sum_{i=1}^{3} \sum_{k=1}^{N_i} \sum_{e_{ik}} \xi_1 \psi_{5i}. \tag{7}$$

If $U \subset H(4, A, {}^1/_2)$, then (5)

$$\psi_{ik}(y_1, y_2) = h_1 \sum_{y_1}^{y_2} \psi_{5i} = O(h_1^{5/2}),$$

where y_1, y_2 are any two points lying on the segment e_{ik}. Consequently, performing in the sum over e_{ik} in expression (7) summation by parts, using the inequality $2ab \leqslant \varepsilon a^2 + \varepsilon^{-1} b^2,\ \varepsilon > 0$, and also estimating $h_1 \sum_{\Gamma_h^1} \xi_1^2$ by

$$h_1^2 \sum_{B_h} \sum_{i=1}^{3} (\xi_{1s_i})^2,$$

we obtain

$$h_1^2 \sum_{B_h} \sum_{i=1}^{3} (\xi_{1s_i})^2 = O(h_1^5),$$

and, taking (3) into account, we obtain

$$h_2 \sum_{l} (\eta'_{1l})^2 = O(h^4),$$

where l is any broken line forming a hexagonal mesh, and η'_l are the divided differences of the function η' along l; hence it follows that $\eta' = O(h_2^2)$ and

$$\eta = O(h_2^2 + \varepsilon)$$

throughout the whole domain Ω.

2) Let $\xi|_{\Gamma_h^1} = \varepsilon_1(h_2)$; then, carrying out arguments analogous to case 1), we obtain that in every interior subdomain of the domain Ω

$$\eta = O(\varepsilon_1(h_2) + h_2^2)$$

and in the whole domain Ω

$$\eta = O\left(\frac{\varepsilon_1(h_2)}{h_2} + h_2^2\right).$$

If one uses the properties of the Green' s functions of the difference problems considered, these estimates can be obtained under weaker assumptions concerning the smoothness of the function U. Analogous error estimates arising in the application of hexagonal meshes can be obtained for elliptic equations with variable coefficients, and also for some nonstationary equations, for example for the heat-conduction equation.

Received
10 XI 1960

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Note: Figure translations are in progress. See original paper for figures.

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