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Abstract

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NECESSARY AND SUFFICIENT CONDITIONS FOR THE UNIFORM EMBEDDING OF A SPACE INTO EUCLIDEAN SPACE

(Presented by Academician P. S. Aleksandrov on 18 II 1961)

In the first part of this note, necessary and sufficient conditions are given for a metric space to be uniformly embedded in a Euclidean space, while in the second these conditions are formulated for uniform spaces*.

1A. We shall call a space R **noncondensing** if there exist a positive number d , a base of uniform coverings** α_i , and a sequence of natural numbers ρ_i , such that the number of elements of the i -th covering α_i that intersect an arbitrarily chosen set M of diameter less than d is not greater than ρ_i . The number*** $\mu = \sup d$ will be called the **modulus of noncondensation**. If $\mu = \infty$, then the space R will be called **absolutely noncondensing**.

It is easy to see that the property of noncondensation is preserved under uniform (in both directions) homeomorphisms. Since every Euclidean space is noncondensing, the property of noncondensation is a necessary condition for uniform embeddability in Euclidean spaces. For uniformly zero-dimensional spaces it is at the same time also sufficient (Theorem 3). In the general case this is not so even for uniformly one-dimensional spaces (see Example 1).

Theorem 1. *A space R of uniform dimension n is uniformly embeddable in Euclidean space E^N , where $N \geq 2n + 1$, if and only if it is noncondensing and if, in addition, one can find a mapping f (generally speaking, discontinuous!) of the space R into E^N , satisfying simultaneously the following conditions:*

ab) there exist numbers $a > 0$ and $b > 0$ such that, if $\rho(x, y) < a$, then $\rho(fx, fy) < b$ for all x and y from R ;

cd) there exist numbers $c > 0$ and d , $0 < d < \mu$, where μ is the modulus of noncondensation of the space R , such that, if $\rho(fx, fy) < c$, then $\rho(x, y) < d$ for all x and y from R ;

moreover, the set of all uniform homeomorphisms of the space R into E^N is everywhere dense in the set of all uniformly continuous mappings satisfying condition cd).

* In the first part of the note, by a space we shall everywhere mean a **metric** —of course, with a countable base—space, and in the second a **uniform** space having a countable base.

** A covering α of the space R is called **uniform** if, for some positive number ε , the covering ω_ε , consisting of the ε -neighborhoods of all points $x \in R$, is inscribed in α . The number $\lambda = \sup \varepsilon$ is called the **Lebesgue number of the covering**. A base of coverings (uniform!) is any system of coverings α_σ such that in every covering one of the coverings α_σ is inscribed. For metric spaces one can always choose a countable base of coverings.

*** The condition of noncondensation is equivalent to the existence of a star-finite base in the sense of Isbell (⁵).

From this follow two important corollaries:

Corollary 1. A space R of uniform dimension $\delta dR = n$ is uniformly embeddable in the space E^N , $N \geq 2n + 1$, if and only if it is non-branching and if, moreover, there exists a uniformly continuous mapping of the space R into E^N satisfying condition cd).

Corollary 2. An absolutely non-branching space R of uniform dimension $\delta dR = n$ is uniformly embeddable in E^N , $N \geq 2n + 1$, if and only if there exists at least one mapping f of the space R into E^N such that for some numbers $a > 0$ and $b > 0$, from $\rho(x, y) < a$ it follows that $\rho(fx, fy) < b$, and from $\rho(fx, fy) < a$ it follows that $\rho(x, y) < b$.

Let us note in this connection that every non-branching space can be metrized uniformly so that it becomes absolutely non-branching (Theorem 2).

1b. The method of proof is a development of the well-known method of Gurevich—Kuratowski (^{1,2}). By \mathfrak{R}^N we shall denote the space of all uniformly continuous mappings of the space R into the space E^N , with the following metric:

$$\rho(f, g) = \sup_{x \in R} \frac{\rho(fx, gx)}{1 + \rho(fx, gx)}.$$

A mapping f of the space R into the space Y will be called an α -mapping, where α is some covering* of the space R , if there is a covering ω of the space Y the full preimage** of which is inscribed in α . By G_α^N we shall denote the set of all α -mappings of the space R into E^N .

Lemma 1. The space \mathfrak{R}^N is complete; the sets G_α^N are open in \mathfrak{R}^N .

Lemma 2. For any base of coverings $\{\alpha_i\}$ of the space R , the intersection

$$\bigcap_i G_{\alpha_i}^N$$

consists of uniform homeomorphisms.

Let M_α^N be the set of all such mappings f from \mathfrak{R}^N for each of which there exist (depending on f) a covering ω of the space E^N and a number p such that the

preimage $f^{-1}U$ of each element U of the covering ω intersects no more than p elements of the covering α . A set $G \subseteq \mathfrak{R}$ will be called **dense relative to** a set $M \subseteq \mathfrak{R}$ if $M \subseteq [G]$.

Lemma 3. If $\delta dR = n$ and $N \geq 2n + 1$, then for any covering α of the space R the set G_α^N is dense relative to the set M_α^N .

Indeed, let $f \in M_\alpha^N$, and let ω be such a covering of the space E^N that the number of elements of the covering α intersecting $f^{-1}U$ does not exceed some fixed number p for each U of ω . Consider the covering η , consisting of all cubes Q_j with centers at all possible vertices of the cubical lattice of the space E^N with step $\varepsilon/2\sqrt{N}$ (the $\varepsilon/2\sqrt{N}$ -cubulation), with edges of length ε/\sqrt{N} , parallel to the coordinate axes, where ε is less than the Lebesgue number of the covering ω . It is inscribed in ω . In the covering^{***} $\alpha \wedge f^{-1}\eta$ we inscribe combinatorially a covering $\gamma = \{\Gamma_i\}$ of multiplicity $\leq n + 1$, and to each of its elements Γ_i we assign a point a_i of E^N such that: 1) $a_i \in Q_{j_i}$, where j_i is the least of all numbers j such that $\Gamma_i \subseteq f^{-1}Q_j$; 2) $\rho(a_i, E^N \setminus Q_{j_i}) > \delta$ for some fixed number $\delta > 0$; 3) the points a_i are in general position. Since $N \geq 2n + 1$, it is possible to realize the nerve of the covering γ in E^N as a complex K' with vertices a_i . Since the lengths of all edges of the complex K' are bounded above by the number 2ε and in

* By a covering we shall everywhere mean a uniform covering.

** That is, the covering consisting of the full preimages of the elements of the covering ω .

*** $\alpha \wedge \beta$ is the product of the coverings α and β , i.e. the totality of all possible intersections $A \cap B$, where $A \in \alpha$, $B \in \beta$.

each cube Q_j contains no more than $2^m p$ vertices of K' , then, by Isbell's lemma⁽⁶⁾, the complex K' can, by an arbitrarily small displacement of its vertices, be transformed into a uniform* complex⁽³⁾ K with vertices b_i such that $b_i \in Q_{j_i}$. The covering of the polyhedron \tilde{K} by the stars of its vertices is uniform⁽⁴⁾; therefore the barycentric mapping g of the space R into \tilde{K} , subordinate to the covering γ , will be uniformly continuous⁽⁴⁾. Finally, $g \in G_\alpha^N$ and $\rho(f, g) < \varepsilon$, as was required to prove.

Lemma 4. If $\delta dR = n$, $N \geq 2n + 1$, and $\{\alpha_i\}$ is some basis of coverings of the space R , then the set $\bigcap_i G_{\alpha_i}^N$ is dense in every open subset of the intersection $\bigcap_i M_{\alpha_i}^N$ in \mathfrak{R}^N .

Indeed, if M is open in \mathfrak{R}^N and $M \subseteq \bigcap_i M_{\alpha_i}^N$, then each open set $G_i = G_{\alpha_i}^N \cap M$ is dense in the closure $[M]$. But $[M]$ is complete. Hence, by Baire's theorem, the intersection $\bigcap_i G_i$ is dense in $[M]$, as was required to prove.

Let now A be some sequence of coverings α_i of the space R . By M_A^N we shall denote the set of all such mappings $f \in \mathfrak{R}^N$ for each of which there exist (depending on f) a covering ω of the space E^N and a sequence of numbers p_i

such that every element of the covering $f^{-1}\omega$ meets no more than p_i elements of the covering α_i .

Lemma 5. For any sequence A of coverings α_i , the set M_A^N is open in \mathfrak{R}^N , and $M_A^N \subseteq \bigcap_i M_{\alpha_i}^N$.

From Lemmas 2, 4, and 5 it follows:

Main lemma. Let $\delta dR = n$ and $N \geq 2n + 1$; if there exists a basis A of coverings α_i of the space R such that the set M_A^N is nonempty, then the space R is uniformly embedded in E^N ; moreover, the set of uniform homeomorphisms of the space R into E^N is dense in the set M_A^N .

On the basis of this, we shall prove only the sufficiency of the conditions of Theorem 1. Let $\delta = \frac{1}{2} \min\{a, d\}$. Into the covering $\omega_\delta = \{O_\delta x\}$, where x ranges over all R , insert a countable covering $\gamma = \{\Gamma_i\}$ of multiplicity $\leq n + 1$ and such that every set of the space R of diameter $< d$ meets no more than p elements Γ_i , where p is fixed. To each Γ_i assign a point a_i in E^N so that: 1) the points a_i are in general position and 2) $\rho(a_i, f\Gamma_i) < c/4$. Since $N \geq 2n + 1$, the nerve of the covering γ is realized in E^N as a complex K' with vertices a_i . Since the lengths of all its edges are bounded above by the number $2b + c$, and in each ball of radius $c/4$ there lie no more than p vertices a_i , the nerve of the covering γ can be realized in E^N as a uniform complex K . If g is the barycentric mapping of the space R into \tilde{K} , subordinate to the covering γ , and A is a basis of coverings α_i corresponding (in the sense of the definition of non-compressibility) to the number d , then $g \in M_A^N$, as was required to prove.

II. The results obtained are generalized to uniform spaces with the natural replacement of the numbers a and d by coverings. For example, the space R is called **non-compressible** if there exists a **separating** covering γ and, for it, a basis of coverings α_i and a sequence of numbers p_i such that the number of elements of the covering α_i meeting an arbitrary element Γ of the covering γ is no greater than p_i . We note that

* A complex is called uniform if the lengths of all its edges are bounded above by some positive number, and at the same time all pairwise distances between its nonintersecting simplexes are bounded below by some positive number ⁽³⁾.

Isbell's embedding theorem ⁽⁶⁾ is a simple consequence of our generalized Theorem 1*.

Theorem 2. *Every nonexpanding space R can be uniformly metrized in such a way that it becomes absolutely nonexpanding.*

Indeed, let $\rho(x, y)$ be some uniform metric of the space R . Taking from the base $\{\alpha_i\}$ a cover $\alpha = \alpha_{j_0}$ inscribed in the refining cover γ , we again obtain a refining cover α such that, for each fixed element $A_{j_0} \in \alpha$ of it, the number of elements $A_j \in \alpha$ intersecting A_{j_0} does not exceed a number p independent of the choice of A_{j_0} . Let ε be the Lebesgue number of the cover α . We shall call a set $M \subseteq R$ ε -connected if, for every pair of points x and y in M , there

exists a chain $x_0 = x, \dots, x_s = y$ such that $x_i \in M$ and $\rho(x_{i-1}, x_i) < \varepsilon$. Then the space R decomposes into a sum of ε -connected sets X_n in such a way that $\rho(X_k, X_n) \geq \varepsilon$ for $k \neq n$. Let $a_n \in X_n$. Let

$$r(x, y) = \inf \sum_{i=1}^s \rho(x_{i-1}, x_i),$$

where $x, y \in X_n$, and the lower bound is taken over all ε -chains connecting x and y . Let $r(a_k, a_n) = |n - k|$, and let

$$r(x, y) = r(x, a_k) + r(a_k, a_n) + r(a_n, y),$$

where $x \in X_k$, $y \in X_n$, and $k \neq n$. Let d be the diameter of the set M . For any cover α_i from the base it is easy to compute that the number of its elements intersecting M is no greater than

$$(d + 1)p_i p^{\frac{2d+\varepsilon}{\varepsilon}},$$

where p_i is the number corresponding to the cover α_i .

Theorem 3. *A uniformly zero-dimensional space is uniformly embeddable in a Euclidean space if and only if it is nonexpanding; moreover, in that case it is embeddable even in a line.*

Example 1.** The space D is an infinite tree, all edges

$$I_0, I_{00}, I_{01}, I_{000}, I_{001}, I_{010}, I_{011}, \dots$$

of which have length 1. The edges $I_{\alpha_1 \dots \alpha_n}, I_{\alpha_1 \dots \alpha_n 0}$ and $I_{\alpha_1 \dots \alpha_n 1}$ have only one common vertex, and the edges have no other points of intersection. Distances between points are measured along broken lines. This is a uniformly one-dimensional, absolutely nonexpanding polyhedron, not uniformly embeddable in any Euclidean space.

Example 2. The one-dimensional skeleton of the cubical decomposition of Euclidean space E^{N+1} is a uniformly one-dimensional polyhedron not uniformly embeddable in E^N .

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CITED LITERATURE

- ¹ W. Hurewicz, Sitzungsber. Preuss. Akad. Wiss., phys.-math. Klasse, H. 24/25, 754 (1933).
- ² C. Kuratowski, Fund. Math., 28, 336 (1937).
- ³ Yu. M. Smirnov, Matem. sborn., 40, No. 2, 137 (1956).
- ⁴ V. I. Egorov, Matem. sborn., 48, No. 2, 227 (1959).
- ⁵ J. R. Isbell, Pacific J. Math., 8, 67 (1958).
- ⁶ J. R. Isbell, Ann. of Math., 70, No. 1, 73 (1959).

* It is easy to obtain also from our main lemma.

** Yu. M. Smirnov drew my attention to this example for another reason.

Note: Figure translations are in progress. See original paper for figures.

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