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Abstract

Full Text

MATHEMATICS

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ON THE SOLVABILITY OF NONLINEAR RITZ SYSTEMS

(Presented by Academician V. I. Smirnov, 24 XII 1960)

The application of the Ritz method to nonlinear problems leads to the necessity of solving nonlinear systems of a finite number of equations with a finite number of unknowns. In the paper of D. F. Davidenko ⁽¹⁾ a method was outlined for solving such systems, consisting in reducing them to systems of ordinary differential equations. An important feature of this method is that one has to solve the Cauchy problem on a finite interval of variation of the independent variable.

In the present note we establish conditions sufficient for the solvability of such a Cauchy problem, to which the Ritz equations lead. Thus the solvability of the Ritz system will be proved.

Consider a functional $f(x)$, defined on a linear set dense in the real Hilbert space H , and which is the potential ⁽²⁾ of some operator $F(x)$.

We shall assume that the Gateaux differential $DF(x, h)$ of the operator $F(x)$ is a uniformly positive operator bounded below. This means that there exists a positive constant γ , independent of x and h , such that the inequality

$$(DF(x, h), h) \geq \gamma|h|^2$$

holds.

Let us pose the problem of the minimum of the functional $f(x)$. We shall solve it by the Ritz method, for which we choose a sequence of coordinate elements $\{x_i\}$ and shall seek an approximate solution in the form

$$x \approx \sum_{i=1}^n a_i x_i, \quad a_i = \text{const.}$$

The Ritz method leads to the following system of equations for the coefficients a_i :

$$\left(Df \left(\sum_{i=1}^n a_i x_i \right), x_j \right) = 0, \quad j = 1, 2, \dots, n,$$

where Df is the Gateaux differential of the functional f .

Since $f(x)$ is the potential of the operator $F(x)$, we have

$$\left(Df \left(\sum_{i=1}^n a_i x_i \right), x_j \right) = \left(F \left(\sum_{i=1}^n a_i x_i \right), x_j \right),$$

and the Ritz system is written in the form

$$\left(F \left(\sum_{i=1}^n a_i x_i \right), x_j \right) = 0, \quad j = 1, 2, \dots, n. \quad (1)$$

The Bubnov–Galerkin method, applied to the equation $F(x) = 0$, leads to this same system (1).

Following D. F. Davidenko ⁽¹⁾, we construct a new system, depending on a certain parameter λ , so that for $\lambda = 0$ the solution of the system is found without difficulty, and for $\lambda = 1$ the transformed system becomes the original one. We take this new system in the form

$$a_j + \lambda \left[\left(F \left(\sum_{i=1}^n a_i x_i \right), x_j \right) - a_j \right] = 0, \quad j = 1, 2, \dots, n; \quad (2)$$

for $\lambda = 0$ we have $a_j = 0$, $j = 1, 2, \dots, n$.

We differentiate the equations of the system with respect to the parameter λ :

$$\frac{da_j}{d\lambda} + \left(F \left(\sum_{i=1}^n a_i x_i \right), x_j \right) - a_j + \lambda \left\{ \sum_{k=1}^n \left[\left(DF \left(\sum_{i=1}^n a_i x_i, x_k \right), x_j \right) - \delta_{jk} \right] \frac{da_k}{d\lambda} \right\} = 0, \quad j = 1, 2, \dots, n. \quad (3)$$

Considering the sum $\sum_{i=1}^n a_i x_i$ as a fixed element, one can introduce the energy product ⁽³⁾

$$\left(DF \left(\sum_{i=1}^n a_i x_i, x_k \right), x_j \right) = [x_k, x_j].$$

Then the matrix of the coefficients of the derivatives $da_k/d\lambda$ can be written in the form

$$(1 - \lambda)E + \lambda \begin{pmatrix} [x_1, x_1] & [x_2, x_1] & \cdots & [x_n, x_1] \\ [x_1, x_2] & [x_2, x_2] & \cdots & [x_n, x_2] \\ \cdots & \cdots & \cdots & \cdots \\ [x_1, x_n] & [x_2, x_n] & \cdots & [x_n, x_n] \end{pmatrix}. \quad (4)$$

This matrix is symmetric, and therefore it can be regarded as the matrix of coefficients of a certain quadratic form. The second matrix in (4) is the Gram matrix of the linearly independent elements x_1, x_2, \dots, x_n , and the corresponding quadratic form is positive definite. The matrix (4) also has the same property, because $0 \leq \lambda \leq 1$. Hence it follows that the determinant Δ_n of the matrix (4) is nonzero for all $\lambda \in [0, 1]$, and the system (3) is solvable with respect to the derivatives $da_j/d\lambda$, which, by Cramer's formulas, can be represented in the form

$$\frac{da_j}{d\lambda} = \frac{\Delta_n^j}{\Delta_n} \equiv g_j(\lambda, a_1, a_2, \dots, a_n), \quad j = 1, 2, \dots, n. \quad (5)$$

We assume the following:

- 1) $(F(\sum_{i=1}^n a_i x_i), x_j)$ and $(DF(\sum_{i=1}^n a_i x_i, x_k), x_j)$ are continuous functions with respect to a_1, a_2, \dots, a_n and have polynomial order of growth:

$$\left(F \left(\sum_{i=1}^n a_i x_i \right), x_j \right) \leq p_m(a_1, a_2, \dots, a_n); \quad (6)$$

$$\left(DF \left(\sum_{i=1}^n a_i x_i, x_k \right), x_j \right) \leq p_{m-1}(a_1, a_2, \dots, a_n). \quad (7)$$

In formulas (6) and (7), p_m and p_{m-1} denote polynomials of degrees m and $m-1$, respectively; m is some natural number.

- 2) The estimate $(DF(\sum_{i=1}^n a_i x_i, h), h)$ from below holds:

$$\left(DF \left(\sum_{i=1}^n a_i x_i, h \right), h \right) \geq N \left(\sum_{i=1}^n a_i^2 \right)^{(m-1)/2} \|h\|^2, \quad N = \text{const}. \quad (8)$$

From estimates (7) and (8) there follows the following estimate for Δ_n :

$$N \left(\sum_{i=1}^n a_i^2 \right)^{n(m-1)/2} \leq \Delta_n \leq p_{n(m-1)}(a_1, a_2, \dots, a_n), \quad (9)$$

where N is some positive constant and $p_{n(m-1)}$ is some polynomial of degree $n(m-1)$.

Indeed,

$$\begin{aligned} \sum_{i,j=1}^n [x_i, x_j] t_i t_j &= \left\| \sum_{i=1}^n x_i t_i \right\|^2 \geq \\ &\geq N \left(\sum_{i=1}^n a_i^2 \right)^{(m-1)/2} \left\| \sum_{i=1}^n x_i t_i \right\|^2 \geq N_1 \left(\sum_{i=1}^n a_i^2 \right)^{(m-1)/2} \sum_{i=1}^n t_i^2, \end{aligned}$$

whence the estimate for Δ_n from below follows.

We now prove that a solution of system (5) exists on the whole interval $0 \leq \lambda \leq 1$. As initial conditions we take the values $a_1 = a_2 = \dots = a_n = 0$, corresponding to the value $\lambda = 0$. Note that the right-hand sides of system (5) are continuous in the domain $0 \leq \lambda \leq 1$, $-\infty < a_j < \infty$; $j = 1, 2, \dots, n$.

For $\lambda = 0$ we have $\Delta_n = 1$. Choose numbers $\delta \in (0, 1)$, $a^* < \infty$, and consider the domain $0 \leq \lambda \leq \delta$, $-a^* \leq a_j \leq a^*$. In this domain all conditions of the existence theorem are satisfied, and, consequently, system (5) has a unique solution satisfying the initial condition and defined on the interval

$$0 \leq \lambda \leq h, \quad h = \min \left\{ \delta, \frac{a^*}{M} \right\}, \quad M = \max |g_j|.$$

Let, for $\lambda = h/2$, $a_1 = \tilde{a}_1, \dots, a_n = \tilde{a}_n$. Next consider the domain

$$h/2 \leq \lambda \leq 1, \quad -\infty < a_j < \infty. \quad (10)$$

We shall prove that in this domain there exists a unique solution of the system satisfying the initial condition

$$\lambda = h/2, \quad a_j = \tilde{a}_j, \quad j = 1, 2, \dots, n. \quad (11)$$

From estimates (6)–(9) it follows that the derivatives $\partial g_j / \partial a_i$ are bounded in domain (10), and then, as is known (see, for example, ⁴), there exists a unique solution of system (5), satisfying the initial conditions (11) and defined on the interval $h/2 \leq \lambda \leq 1$.

It is now clear that system (3) has a solution defined on the entire interval $0 \leq \lambda \leq 1$ and satisfying the initial conditions $\lambda = 0$, $a_j = 0$, $j = 1, 2, \dots, n$; from this follows the solvability of the Ritz system (1).

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CITED LITERATURE

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² M. M. Vainberg, *Variational Methods for the Study of Nonlinear Operators*, 1956.

³ S. G. Mikhlin, *Variational Methods in Mathematical Physics*, 1957.

⁴ J. Sansone, *Ordinary Differential Equations*, 1, IL, 1953.

Note: Figure translations are in progress. See original paper for figures.

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