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Abstract

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NEW CRITERIA FOR PARACOMPACTNESS AND METRIZABILITY OF AN ARBITRARY T_1-SPACE

(Presented by Academician P. S. Aleksandrov, 31 V 1961)

This and my preceding paper, devoted to the problem of metrization of spaces, are closely connected with the recent works of P. S. Aleksandrov ([1]) and V. I. Ponomarev ([2]), in which a new approach to the metrization problem was found, one that proved to be very fruitful.

One of the possible equivalent definitions of the concept of a complete system of coverings (in the sense of P. S. Aleksandrov and P. S. Uryson ([3])) is the following. A system φ of coverings γ^* of a topological space X is called **complete** if, for every point $x \in X$ and an arbitrary neighborhood $Ox \ni x$ of it, there exists such a $\gamma \in \varphi$ that every element of the covering γ containing the point x is contained in Ox, or, denoting by $\gamma(A)$ the star of the set A in the covering γ^{**}, we may write the condition just formulated as follows:

$$\gamma(x) \subseteq Ox.$$

Thus, the concept of a complete system of coverings in the sense of P. S. Aleksandrov and P. S. Uryson coincides with the concept (introduced considerably later) of a refining system of coverings.

The concept of an α-base of a topological space introduced below is a strengthening of the concept of a countable complete system of coverings.

Definition 1. A base B of a topological space X is called an α-base (of this space) if it can be decomposed into a countable set $\varphi = \{\gamma_i\}$ of coverings γ_i of the space X in such a way that for every point $x_0 \in X$ and every neighborhood Ox_0 of it there exist a smaller neighborhood $O_1x_0 \ni x_0$ and a covering $\gamma_n \in \varphi$ for which

$$\gamma_n(O_1x_0) \subseteq Ox_0$$

holds***.

How broad the concept of an α-base is can be seen from the fact that every base of any metric space is an α-base (which is easily verified).

It is now convenient for us to introduce:

Definition 2. A sequence $\varphi = \{\gamma_i\}$ of open coverings is **majorized** by a covering ξ if, for every point $x \in X$, for some n, Ox, and $G_\alpha \in \xi$, one has

$$\gamma_n(Ox) \subseteq G_\alpha.$$

It is natural to call a sequence of coverings of a certain space **fundamental** if it is majorized by every covering of this space.

The main results of this note are Theorems 1 and 2.

Theorem 1. *For paracompactness**** of a space X it is necessary and sufficient that, for every covering ξ of this space, there be found*

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* In this note, by coverings we shall mean only open coverings, without specifying this each time.

** That is, the union of the elements of the covering γ that intersect A.

*** In particular, condition (α) is fulfilled if, for an arbitrary point $x \in X$, the collection of sets

$$\gamma_i^2(x) = \gamma_i(\gamma_i(x)), \quad \text{where } i = 1, 2, \dots, k, \dots,$$

forms a base at this point.

**** Here and in what follows, by paracompactness of the space X it is convenient to mean the possibility of star-refining into any open covering ξ of the space X some open covering η.

···there was a sequence $\varphi = \{\gamma_i\}$ of coverings of the same space, majorized by the covering ξ.

Theorem 2. In order that a T_0-space be metrizable, it is necessary and sufficient that it possess an α-base.

Before proceeding to the proofs of Theorems 1 and 2, let us note at once that if there exists an α-base in a space X, then the sequence of coverings that defines this α-base is majorized by any covering of our space, i.e. it forms a fundamental sequence of coverings of the space X. It is easy to verify that in the case of T_1-spaces the converse is also true: the elements of every fundamental system of coverings of some space form an α-base of this space.

Without the assumption of the T_1 separation axiom, as the example of the connected two-point space shows, this need no longer be so.

Theorem 2 admits the following formulation:

Theorem 2′. In order that a T_1-space be metrizable, it is necessary and sufficient that there exist in it a fundamental sequence of coverings.

We shall show that if in a space there exists an α-base and the T_0-separation axiom is satisfied, then the Hausdorff separation axiom is also satisfied in it.

Consider an arbitrary pair of points x and y of the space X. At least for one of them, say for definiteness for x, there is a neighborhood Ox not containing the other point y, i.e. $Ox \not\ni y$. Let now O_1x and n be chosen so that $\gamma_n(O_1x) \subseteq Ox$. Since γ_n is an open covering of the space X, in γ_n there is an element G_α containing the point y. Obviously $O_1x \cap G_\alpha = \Lambda$, which is what we wanted.

Now Theorems 2 and 2′ are easily reduced to Theorem 1 and to V. Ponomarev' s criterion ([2]): from Theorem 1 it will follow that a space possessing an α-base is paracompact; on the other hand, every α-base decomposes into a complete countable system of coverings. V. Ponomarev' s criterion asserts that in this case the space X is metrizable. Moreover, after Theorem 1 has been proved, Theorems 2 and 2′ are easily reduced also to the first metrization criterion of Aleksandrov-Uryson, and to the well-known criterion of Nagata-Smirnov, and to the regular metrization criterion ([4]).

Proof of Theorem 1. Let $\xi = \{G_\alpha\}$ be an arbitrary covering of the space X, and let $\varphi = \{\gamma_i\}$ be a sequence of coverings of this space majorized by the covering ξ. Obviously, without loss of generality, we may assume that γ_m is inscribed in γ_n, whenever $m > n$.

We shall call an open set $H \subseteq X$ **marked** if there exists a natural number n, a $\Gamma \in \gamma_n$, and a $G_\alpha \in \xi$ such that

$$H \subseteq \Gamma \in \gamma_n, \qquad \gamma_n(H) \subseteq G_\alpha. \tag{1}$$

To each marked H we assign some $n = n(H)$ satisfying only the just-formulated condition.

From the fact that the sequence $\varphi = \{\gamma_i\}$ is majorized by the covering ξ, it follows directly that the totality of all marked sets forms an open covering η of the space X. Let us verify, for an arbitrary point $x \in X$, that $\eta(x)$—its star with respect to the covering η—is contained in some element of ξ.

There is an $H^\alpha \in \eta$ such that $H^\alpha \ni x$, and if $x \in H^\beta$, where $H^\beta \in \eta$, then $n(H^\alpha) \leq n(H^\beta)$. Then

$$\eta(x) \subseteq \bigcup_{i=n(H^\alpha)}^{\infty} \gamma_i(x) \subseteq \gamma_{n(H^\alpha)}(x) \subseteq \gamma_{n(H^\alpha)}(H^\alpha),$$

and, since by construction of the covering η, for some $G \in \xi$ it is true that $\gamma_{n(H^\alpha)}(H^\alpha) \subseteq G$, ultimately $\eta(x) \subseteq G$. We have constructed a covering η, star-refined into the covering ξ, by which Theorem 1, and with it Theorems 2

and 2′ have been proved. Note that along the way we have once again proved the paracompactness of metric spaces.

We shall show how various known metrization conditions can be obtained from the α-criterion.

1. A regular space with a countable base is metrizable. Let B be a countable base in X. For each pair (U_i, U_j), $U_i \in B$, $U_j \in B$, where $U_i \supset [U_j]$, take the cover $\gamma_{i,j} = \{U_i, X \setminus [U_j]\}$. The set of selected covers is countable and obviously forms an α-base in X.

Remark. On the regular metrization criterion.

If the selected covers $\gamma_{i,j}$ are renumbered as: $\gamma_1, \gamma_2, \dots, \gamma_k, \dots$, and then the system of finite covers $\varphi = \{\omega_n\}$ is considered, where $\omega_n = \bigcap_{i=1}^{n} \gamma_i$, then

$$B' = \bigcup_{n=1}^{\infty} \omega_n$$

turns out to be a regular base of the space X (this follows at once from the completeness of the system φ, from the fact that each subsequent cover is inscribed in the preceding ones, and from the finiteness of each cover). Thus, the metrization theorem for T_1-spaces with a regular base ([4]) easily also includes the case of regular spaces with a countable base.

2. (Bing' s theorem). A regular space possessing a σ-discrete base is metrizable. Indeed, let

$$B = \bigcup_{k=1}^{\infty} \gamma'_k,$$

where the γ'_k are discrete systems of open sets in X. For each pair i, j take the open cover $\gamma_{i,j}$ of the space X, consisting of all elements of the cover γ'_i that contain the closures of some elements of γ'_j, and of the complement in X to the sum of the closures of all elements of the cover γ'_j that are contained with their closure in some elements of γ'_i. The totality of the selected covers $\varphi = \{\gamma_{i,j}\}$ is countable and, as may be checked, forms an α-base of the space X. Now, just as in the preceding item, one could construct a regular base in the space X.

3.* For the metrizability of a separable topological group it is necessary and sufficient that it have the first axiom of countability.

Indeed, let $\eta = \{V_i,\ i = 1, 2, \dots, k, \dots\}$ be a defining system of neighborhoods of the identity of the group G. Put $\gamma_i = \{V_i x :\ x \in G\}$. The base

$$B = \bigcup_{i=1}^{\infty} \gamma_i$$

is an α-base of the space of the group G.

4. We have proved the α-criterion of metrizability, relying on the theorem of V. Ponomarev or, what amounts to the same thing, on the Alexandroff-Urysohn theorem. The latter, in turn, is simply a special case of the

α-criterion, for every regular chain in a space, i.e. a countable complete sequence of covers, each subsequent one being properly (star-)inscribed in the preceding one, forms an α-base of this space.

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CITED LITERATURE

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[3] P. S. Urysohn, *Works on topology and other areas of mathematics*, Moscow-Leningrad, 2, 1951, p. 964.
[4] A. Arhangel' skii, Bull. Acad. Polon. Sci., Ser. Math., Astr. et Phys., **8**, 589 (1960).

* This was proved by Markov and Kakutani in the 1940s.

Note: Figure translations are in progress. See original paper for figures.