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Formulas of mechanical cubatures

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Abstract

Full Text

MATHEMATICS

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ON FORMULAS OF MECHANICAL CUBATURES IN n-DIMENSIONAL SPACE

Formulas of mechanical cubatures

$$(l, \varphi) \equiv \int\limits_{\Omega} \varphi\, dx - \sum_{k=1}^{N} C_k \varphi(x^{(k)}) \cong 0, \tag{1}$$

where x is a point of a bounded n-dimensional domain Ω; C_k are coefficients; $x^{(k)}$ are the nodes of the formula, give different accuracy for different classes of functions. We shall assume that the error (l, φ) is equal to zero for polynomials of a certain degree m_1. The domain Ω will be assumed to have a piecewise smooth boundary.

Of special interest for the theory is the particular case of cubature formulas when the function φ is periodic with periods $H\beta$, where H is the fundamental matrix of periods

$$H = (\mathbf{h}_1, \mathbf{h}_2, \ldots, \mathbf{h}_n), \tag{2}$$

each period $\mathbf{h}_k$ is a column vector

$$\mathbf{h}_k = \begin{pmatrix} h_{1k} \\ h_{2k} \\ \ldots \\ h_{nk} \end{pmatrix}; \tag{3}$$

β is an integer column vector

$$\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \ldots \\ \beta_n \end{pmatrix}, \qquad -\infty < \beta_k < +\infty. \tag{4}$$

In this case the domain of integration Ω_0 is the fundamental parallelepiped such that the system of all Ω_β, obtained by translating Ω_0 by $H\beta$, covers the whole space R_n without intersection. In this case we put $m_1 = 0$, i.e. we shall assume that formula (1) is valid for $\varphi = 1$, and hence

$$\sum_{k=1}^{N} C_k = |\Omega_0|. \tag{5}$$

The principal problem of the theory of mechanical cubatures is the determination of

$$\min_{C_k, x^{(k)}} [\max |(l, \varphi)|] = d(X, N) \tag{6}$$

for a given class X and a given number of points N. The values C_k and $x^{(k)}$ for which this minimax is attained give the optimal formula of mechanical cubatures. For $n = 1$ this problem was considered by S. M. Nikol' skii and his students. A survey of the results obtained is given in (^{3}); the most important bibliography of the question is also included there.

Recently, cubature formulas for $n > 1$ have been studied by many authors ($^{7-13}$).

It is convenient to take as X the unit ball in some Banach space B, in which (l, φ) is linear. Let us note that in the space C of continuous functions in Ω or Ω_0 there are no functionals of the form (l, φ) that are small on the whole unit ball, although all of them are linear:

$$\sup_{\|\varphi\|_C = 1} |(l, \varphi)| = \left[|\Omega_0| + \sum_{k=1}^{N} |C_k| \right]. \tag{7}$$

Therefore, in C the main problem has no meaning.

In the present note we shall study the first half of the main problem, i.e. the finding of

$$\max_X |(l, \varphi)| = d(C_k, x^{(k)}) \tag{8}$$

in the case when X is the unit ball in the space of functions whose derivatives of order m are square-integrable. The second half of the problem—the finding of $\min d(C_k, x^{(k)})$—is a problem on the minimum of a function of $(n+1)N$ variables, and we shall not touch upon it. At the same time we shall consider the spaces $\widetilde{W}_2^{(m)}$ and $\widetilde{L}_2^{(m)}$ of functions periodic in R_n, with periods $H\beta$, and for Ω we take the parallelepiped Ω_0.

The norms in $W_2^{(m)}$ and $\widetilde{W}_2^{(m)}$ are given by the formulas (^{2})

$$\|\varphi\|^2_{W_2^{(m)}} = \|\Pi\varphi\|^2_{S_{m-1}} + \|\varphi\|^2_{L_2^{(m)}} = \|\Pi\varphi\|^2_{S_{m-1}} + D(\varphi), \tag{9}$$

$$\|\varphi\|^2_{\widetilde{W}_2^{(m)}} = \left(\int_{\Omega_0} \varphi\, dx\right)^2 + D(\varphi), \tag{10}$$

where Π is the projection operator from $W_2^{(m)}$ into the space S_{m-1} of polynomials of degree m, and $L_2^{(m)}$ is the factor space $W_2^{(m)}/S_{m-1}$. Here

$$\|\varphi\|^2_{L_2^{(m)}} = D(\varphi) = \int_\Omega \sum_{|\alpha|=m} (D^\alpha \varphi)^2\, dx. \tag{11}$$

For the nonperiodic case we shall henceforth always assume $m_1 = m - 1$, i.e.

$$(l, \varphi) = 0 \quad \text{for } \varphi \in S_{m-1}. \tag{12}$$

If the operator $\Pi\varphi$ is interpolatory, then the general case reduces to this one.

The following inequalities hold:

$$|(l,\varphi)| \le K\|\varphi\|_{L_2^{(m)}} \le K\|\varphi\|_{W_2^{(m)}}, \qquad |(l,\varphi)| \le K\|\varphi\|_{\widetilde{L}_2^{(m)}} \le K\|\varphi\|_{\widetilde{W}_2^{(m)}}. \tag{13}$$

Consider three problems:

Problem I. Find

$$\max_{\|\varphi\|_{W_2^{(m)}}=1} (l, \varphi).$$

Problem II. Find

$$\min_{(l,\varphi)=1} \|\varphi\|^2_{W_2^{(m)}}.$$

Problem III. Find

$$\min H_\lambda(\varphi) = D(\varphi) + 2\lambda(l, \varphi).$$

All three problems reduce to any one of them.

Let us consider Problem III by means of the direct method. In view of (9), $H_\lambda(\varphi)$ has a finite exact lower bound:

$$H_\lambda(\varphi) \ge [\sqrt{D(\varphi)} - \lambda K]^2 - \lambda^2 K^2 \ge -\lambda^2 K^2. \tag{14}$$

The identity

$$\frac{1}{2} H_\lambda(u_k) + \frac{1}{2} H_\lambda(u_m) - H_\lambda\left(\frac{u_k + u_m}{2}\right) = D\left(\frac{u_k - u_m}{2}\right) \tag{15}$$

allows one to conclude that if u_k is a minimal sequence, then Πu_k will again be minimal and, moreover, fundamental, with the unique limit giving the solution of problem III. Further, from the identity

$$H_\lambda(\varphi) = \frac{\lambda^2}{\lambda_1^2} H_{\lambda_1}\left(\frac{\lambda_1 \varphi}{\lambda}\right) \tag{16}$$

one may conclude that the solutions of problem III for different λ differ from one another by a factor and are expressed by the formula

$$u_\lambda = \lambda u_1. \tag{17}$$

Consideration of the function $\psi(\mu) = H(\mu u_\lambda)$ leads to the conclusion that if

$$H_\lambda(u_\lambda) = \min H_\lambda(u) = -d_\lambda(C_k, x^{(k)}), \tag{18}$$

then

$$D(u_\lambda) = d_\lambda(C_k, x^{(k)}); \qquad (l, u_\lambda) = -d_\lambda(C_k u^{(k)}). \tag{19}$$

Obviously, when $(l, u_\lambda) = 1$ the solution of III will also be a solution of II, while the solutions of I and II differ by a factor. Hence we obtain the solutions of problems I and II in the form

$$u_I = u_1/d_1; \qquad u_{\mathrm{II}} = -u_1/\sqrt{d_1}. \tag{20}$$

Problems I, II, and III are solved analogously for the periodic case.

The finding of the extremal function can now be reduced to the integration of a partial differential equation. It is convenient to use the apparatus and symbolism of the theory of generalized functions (4–6).

By the classical method of the calculus of variations one obtains the equation in variations for the solution of problem III for $\lambda = 1$:

$$2D(u_1, \xi) - 2\int \xi\, dx - 2\sum C_k \xi(x^{(k)}) = 0, \tag{21}$$

where

$$D(u_1, \xi) = \int \sum_{|\alpha|=m} D^\alpha u_1 D^\alpha \xi\, dx,$$

ξ is an admissible variation (any function from $W_2^{(m)}$ or $\widetilde{W}_2^{(m)}$). We rewrite equation (21) in the form

$$D(u_1,\xi)=\int_\Omega\left[1-\sum_{k=1}^{N}C_k\delta(x-x^{(k)})\right]\xi(x)\,dx \tag{22}$$

($\delta(x)$ is the Dirac function).

For the function u_1, by a slight modification of the classical arguments, we obtain the equation

$$\Delta^m u_1=(-1)^m\left[1-\sum_{k=1}^{N}C_k\delta(x-x^{(k)})\right]; \tag{23}$$

and the boundary conditions

$$B_k(u_1)\big|_S=0 \tag{24}$$

for the nonperiodic case. In the periodic case there are no boundary conditions.

In the nonperiodic case, from (23) it follows that

$$u_1=\frac{\Gamma(n/2)2^{-2m}}{\Gamma(n/2+m)\Gamma(m+1)}r_k^{2m}-$$

$$-\sum_{k=1}^{N}C_k\frac{i^{n+1}2^{-2m+1}\pi^{n/2+1}}{\Gamma(m+n/2+1)\Gamma(m)}r_k^{n-2m}\times\begin{cases}1, & (n\text{ odd});\\ \lg r/2\pi i+u_1^*, & (n\text{ even}),\end{cases} \tag{25}$$

where $r_k=|x-x^{(k)}|$, u_1^* is a solution of the polyharmonic equation $\Delta^m u_1^*=0$, chosen so as to satisfy (24).

In practice, to find u_1^* one may use the method of integral equations, the grid method, or some other device. It is also convenient to apply some direct method, such as, for example, Ritz' s method, for the direct determination of u_1. Let us also consider the periodic case. For simplicity let $|\Omega_0|=1$. Denote by $u^{(k)}$ the periodic solution of the equation

$$\Delta^m u^{(k)}=(-1)^m\,[1-\delta(x-x^{(k)})];\qquad x\in\Omega_0,\quad x^{(k)}\in\Omega_0. \tag{26}$$

Then

$$u_1=\sum C_k u^{(k)}. \tag{27}$$

Let x and y be coordinate column vectors, and let $x = Hy$. The periodic function $\Lambda(x)$ with periods $H\beta$, equal to $1 - \delta(x - x^{(k)})$ in the parallelepiped Ω_0, will be a periodic function $M(y)$ with integer periods β. Its value in the fundamental cube will be $M(y) = [1 - \delta(y - y^{(k)})]$, $-1/2 < y \leq 1/2$. This function is expanded in a generalized Fourier series

$$M(y) = \sum_{|\gamma| \neq 0} \exp[2\pi i(\gamma, y - y^{(k)})], \tag{28}$$

where $\gamma(\gamma_1, \gamma_2, \dots, \gamma_n)$ runs through all possible integer values, both positive and negative. Passing to the variables x, we shall have

$$\Lambda(x) = \sum_{|\gamma| \neq 0} \exp\left[2\pi i(\gamma, H^{-1}(x - x^{(k)}))\right] = \sum_{|\gamma| \neq 0} \exp\left[2\pi i(\gamma H^{-1}, x - x^{(k)})\right], \tag{29}$$

whence

$$u^{(k)} = \sum_{|\gamma| \neq 0} \frac{\exp\left[2\pi i(\gamma H^{-1}, x - x^{(k)})\right]}{(\gamma H^{-1})^{2m}}. \tag{30}$$

Formulas (27), (30) make it easy to compute the desired maximum (l, φ).

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Note: Figure translations are in progress. See original paper for figures.

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