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Abstract

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TWO CLASSES OF INEQUALITIES FOR SUFFICIENTLY SMOOTH FUNCTIONS OF n VARIABLES

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1. In the paper (1) the notions of dimension and differential order of functionals of the type of norms in W_p^l were introduced. It was also noted there that the specificity of the limiting exponents in S. L. Sobolev' s embedding theorems consists in the fact that the corresponding inequalities are sharp with respect to dimension, but at the same time they contain a coarsening with respect to differential order (the function is estimated through its derivatives).

Here we shall single out a class of inequalities that are sharp both with respect to dimension and with respect to differential order. Up to the present time, apparently, only one nontrivial inequality of this type was known. Namely, for $l = \alpha l_1 + (1-\alpha) l_2$; $\frac{1}{p} = \frac{\alpha}{p_1} + \frac{1-\alpha}{p_2}$; $l_2 > l_1$; $p_2 > 1$ and $\frac{l_2 - l_1 - 1}{p} < l_2 - l + \frac{l - l_1 - 1}{p_2}$ for integral l, the relation holds

$$\|u\|_{W_p^l} \le C \|u\|^{\alpha}_{W_{p_1}^{l_1}} \|u\|^{1-\alpha}_{W_{p_2}^{l_2}}. \tag{1}$$

In the particular case when one of the spaces $W_{p_i}^{l_i}$ is contained wholly in the other, relation (1) was found by V. P. Il' in (2). Without this restriction it was proved by Gagliardo (3) and Nirenberg (4).

We shall give three more inequalities, sharp with respect to differential order and dimension.

Theorem 1. *If the conditions are satisfied:* $p, p', p'' > 1$; $\frac{1}{p} = \frac{\alpha}{p'} + \frac{1-\alpha}{p''}$; $r_i = \alpha r_i' + (1-\alpha) r_i''$, $i = 1, \dots, n$, *then*

$$\|u\|_{H_p^{(r_1 \dots r_n)}} \le C \|u\|^{\alpha}_{H_{p'}^{(r_1' \dots r_n')}} \|u\|^{1-\alpha}_{H_{p''}^{(r_1'' \dots r_n'')}}. \tag{2}$$

Here by the norm $\|u\|_{H_p^{(r_1 \dots r_n)}}$ we mean

$$\sum_{i=1}^{n} \max_{h} \left\| \frac{\Delta^2_{h,i} u}{h^{r_i - \bar{r}_i}} \right\|_{L_p(E_n)};$$

$\bar{r}_i$ is the nearest integer less than r_i; $\Delta^2_{h,i} u$ is the finite difference of second order with step h in the i-th coordinate.

Theorem 2. *Under the hypotheses of the preceding theorem the inequality is valid*

$$\|u\|_{\mathcal{L}_p^{(r_1 \dots r_n)}} \leq C \|u\|^{\alpha}_{\mathcal{L}_{p'}^{(r'_1 \dots r'_n)}} \|u\|^{1-\alpha}_{\mathcal{L}_{p''}^{(r''_1 \dots r''_n)}}. \tag{3}$$

Here by the norm $\|u\|_{\mathcal{L}_p^{(r_1 \dots r_n)}}$ we mean

$$\sum_{i=1}^{n} \left[\int_0^{\infty} \frac{dh}{h^{1+(r_i - \bar{r}_i)p}} \int_{E_n} |\Delta^2_{h,i} u^{\bar{r}_i}|^p \, dx \right]^{1/p}.$$

These norms were recently introduced by V. A. Solonnikov; moreover, for fractional and equal $r_i = r$ they are equivalent to the well-known norms of L. N. Slobodetskii W_p^r.)

Theorem 3. If the conditions $l = \alpha r_i + (1-\alpha) r'_i$, $i = 1, \dots, n$; $p > 1$; l is an integer, are satisfied, then

$$\|u\|_{W_p^l(E_n)} \leq C \|u\|^{\alpha}_{H_p^{(r_1 \dots r_n)}} \|u\|^{1-\alpha}_{H_p^{(r'_1 \dots r'_n)}}. \tag{4}$$

A common feature of inequalities (1)–(4) is that the dimension of the space n does not enter into the relation between the quantities α, p, l, and r_i. Moreover, if in (2)–(4) one sets $r_i = r$, $r'_i = r'$, and $r''_i = r''$, then the constants in (1)–(4) also do not depend on n. It can be shown that these facts apply to all inequalities that are sharp with respect to differential order and with respect to dimension. There is a rule by which such an inequality, once proved for $n = 1$, is automatically transferred to the case of arbitrary n.

In proving an inequality that is sharp with respect to differential order and with respect to dimension, a special choice of means is necessary, one that does not lead to a coarsening in these respects. It is inadmissible, for example, to use the fact that

$$|u(b) - u(a)| \leq \int_a^b |u_x| \, dx.$$

Our proofs of Theorems 1 and 2 are based on propositions on the equivalence of a certain smoothness of $u(x)$ to a definite rate of decrease, in the L_p norm, of a finite difference of $u(x)$ of some order when its step tends to zero (see (^{5})).

In inequalities (2) and (3) one may pass from the norms $H_p^{(r_1 \dots r_n)}$, $\mathcal{L}_p^{(r_1 \dots r_n)}$ to equivalent norms such that the sharp constants in the corresponding inequalities are equal to one. Then one may say, using the terminology of S. G. Krein (^{6}), that the spaces $H_p^{(r_1 \dots r_n)}$ and $\mathcal{L}_p^{(r_1 \dots r_n)}$ form scales in each of the upper indices and in the reverse lower one. Next an inequality is established for arbitrary scales of Banach spaces, from which, in particular, generalizations of (2) and (3) follow.

2. Let E_α be some scale of Banach spaces corresponding to the interval $[ab]$ of variation of the parameter α. It is known that if an element u belongs to a linear set M contained in all spaces of the scale, then

$$\log \|u\|_{E_\alpha} \leq h \log \|u\|_{E_\beta} + (1-h) \log \|u\|_{E_\gamma} \tag{5}$$

for $\alpha = h\beta + (1-h)\gamma$; $\alpha, \beta, \gamma \in [ab]$, $0 \leq h \leq 1$.

Inequality (5) can be generalized. Namely, the following theorem, proved on the basis of (5), is valid:

Theorem 4. Let $\Phi_1(\alpha)$ and $\Phi_2(\alpha)$ be nondecreasing, left-continuous functions with unit variation on the interval $[ab]$ and such that

$$\int_a^b \alpha \, d\Phi_1(\alpha) = \int_a^b \alpha \, d\Phi_2(\alpha).$$

Then, in order that for every $u \in M$ the inequality

$$\int_a^b \log \|u\|_{E_\alpha} \, d\Phi_1(\alpha) \leq \int_a^b \log \|u\|_{E_\alpha} \, d\Phi_2(\alpha), \tag{6}$$

be satisfied, it is sufficient that each equality

$$\int_a^{\beta_1} d\Phi_1(\alpha) + \gamma_1[\Phi_1(\beta_1)] = \int_a^{\beta_2} d\Phi_2(\alpha) + \gamma_2[\Phi_2(\beta_2)], \qquad 0 \leq \gamma_i \leq 1, \tag{7}$$

was accompanied by the inequality

$$\int_a^{\beta_1} \alpha \, d\Phi_1(\alpha) + \gamma_1\beta_1[\Phi_1\beta_1] \geq \int_a^{\beta_2} \alpha \, d\Phi_2(\alpha) + \gamma_2\beta_2[\Phi_2(\beta_2)]. \tag{8}$$

It turns out that the condition of this theorem is not only sufficient, but also necessary for relation (6) to hold, if the scale E_α satisfies a certain special condition. The latter is as follows:

A. M contains a two-parameter family of elements $u_{\gamma,\varepsilon}$, where $\gamma \in [ab]$, and ε is sufficiently small, such that

$$\log \|u_{\gamma,\varepsilon}\|_{E_a} = (a-\gamma)^+ \log \frac{1}{\varepsilon} + C(a,\gamma,\varepsilon),$$

where

$$|C(a,\gamma,\varepsilon)| < \text{const} \cdot \psi(\varepsilon), \qquad \psi(\varepsilon) \Big/ \log \frac{1}{\varepsilon} \to 0 \quad \text{as } \varepsilon \to 0.$$

Theorem 5. Let the scale E_α satisfy condition A, and suppose that for every $u \in M$ inequality (6) holds, where the functions $\Phi_i(\alpha)$ are nondecreasing, continuous from the left, have unit variation, and satisfy the equality

$$\int_a^b \alpha \, d\Phi_1(\alpha) = \int_a^b \alpha \, d\Phi_2(\alpha).$$

Then from equality (7) there follows inequality (8).

To prove this theorem one should apply inequality (6) to the functions $u_{\beta_2,\varepsilon}$ and compare the orders of growth of both sides as $\varepsilon \to 0$. This will lead to the inequality

$$\int_a^b (\alpha - \beta_2)^+ \, d\Phi_1(\alpha) \le \int_a^b (\alpha - \beta_2)^+ \, d\Phi_2(\alpha).$$

From this, and from equality (7), taking into account the remaining conditions satisfied by $\Phi_i(\alpha)$, relation (8) follows.

One can give a number of examples of scales satisfying condition A. In particular, all the scales indicated by S. G. Krein in paper (^{6}) satisfy it. We shall give two concrete theorems based on the preceding results.

Theorem 6. In order that, for every bounded or sufficiently highly integrable function $u(x)$ given in some domain Ω of n-dimensional space, the inequality

$$\int_0^\infty \log \|u\|_{L_p} \, d\Phi_1(p) \le \int_0^\infty \log \|u\|_{L_p} \, d\Phi_2(p),$$

where $\Phi_1(p)$ and $\Phi_2(p)$ are nondecreasing functions continuous from the left, with unit variation, such that

$$\int_0^\infty \frac{1}{p}\, d\Phi_1(p) = \int_0^\infty d\Phi_2(p),$$

it is necessary and sufficient that each equality (for $0 \le \gamma_i \le 1$)

$$\int_0^{\beta_1} d\Phi_1(p) + \gamma_1[\Phi_1(\beta_1)] = \int_0^{\beta_2} d\Phi_2(p) + \gamma_2[\Phi_2(\beta_2)]$$

be accompanied by the inequality

$$\int_0^{\beta_1} \frac{1}{p}\, d\Phi_1(p) + \frac{\gamma_1}{\beta_1}[\Phi_1(\beta_1)] \le \int_0^{\beta_2} \frac{1}{p}\, d\Phi_2(p) + \frac{\gamma_2}{\beta_2}[\Phi_2(\beta_2)].$$

The sufficiency of the condition of the theorem follows from the fact that the norms in $L_{n/\alpha}$ satisfy (5) for $0 \le \alpha \le \infty$. The necessity follows from the fact that satis-

condition A is satisfied; moreover, one may set:

$$u_{n/p,\varepsilon}(x) = \begin{cases} \dfrac{1}{|x-y|^p}, & \text{for } \varepsilon \le |x-y| \le b; \\ 0, & \text{for } |x-y| < \varepsilon \text{ and } |x-y| > b, \end{cases}$$

where y is any interior point of Ω; $b \equiv \min\{1, \rho_y\Gamma\}$; $\rho_y\Gamma$ is the distance from y to the boundary Γ.

Theorem 7. In order that, for every sufficiently smooth function $u(x)$ defined in some domain Ω of n-dimensional space, the inequality

$$\int_0^1 \log \|u\|_{\mathrm{Lip}\,\alpha}\, d\Phi_1(\alpha) \le \int_0^1 \log \|u\|_{\mathrm{Lip}\,\alpha}\, d\Phi_2(\alpha),$$

hold, where $\Phi_i(\alpha)$ are nondecreasing, left-continuous functions with unit variation such that

$$\int_0^1 \alpha\, d\Phi_1(\alpha) = \int_0^1 \alpha\, d\Phi_2(\alpha),$$

it is necessary and sufficient that every equality (for $0 \le \gamma_i \le 1$)

$$\int_0^{\beta_1} d\Phi_1(\alpha) + \gamma_1[\Phi_1(\beta_1)] = \int_0^{\beta_2} d\Phi_2(\alpha) + \gamma_2[\Phi_2(\beta_2)]$$

be accompanied by the inequality

$$\int_0^{\beta_1} \alpha\, d\Phi_1(\alpha) + \gamma_1\beta_1[\Phi_1(\beta_1)] \geq \int_0^{\beta_2} \alpha\, d\Phi_2(\alpha) + \gamma_2\beta_2[\Phi_2(\beta_2)].$$

The sufficiency of the condition of the theorem follows from the fact that the spaces Lip α form a scale. The necessity follows from the fact that condition A is satisfied; moreover, using the notation of the preceding theorem, one may set:

$$u_{\gamma,\varepsilon}(x) = \begin{cases} |x-y|^\gamma, & \text{for } \varepsilon \leq |x-y| \leq b; \\ \varepsilon^\gamma, & \text{for } |x-y| < \varepsilon; \\ b^\gamma, & \text{for } |x-y| > b. \end{cases}$$

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REFERENCES

1. K. K. Golovkin, DAN, **134**, No. 1 (1960).
2. V. P. Il’ in, Tr. Matem. Inst. im. V. A. Steklova AN SSSR, **53**, 64 (1959).
3. E. Gagliardo, Rich. di Math., **7**, F. 1, 102 (1958).
4. L. Nirenberg, Ann. di Scuola norm. sup., S. 3, **13**, F. 2, 1 (1959).
5. K. K. Golovkin, DAN, **134**, No. 6 (1960).
6. S. G. Krein, DAN, **130**, No. 3 (1960).

Note: Figure translations are in progress. See original paper for figures.

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