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Abstract

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LINEARLY ENVELOPING COUPLER CURVES OF MECHANISMS WITH A PRISMATIC PAIR

The present study is devoted to the theory of linearly enveloped coupler curves of mechanisms with three revolute pairs and one prismatic pair.

Consider the slider-crank mechanism ABC (Fig. 1). In the plane of the connecting rod choose an arbitrary straight line $u—u$. As was shown earlier ([1]), it may be assumed that this straight line passes through point B at an angle γ to the connecting rod BC. Drop from point A the perpendiculars AF and AE to the straight lines $u—u$ and BC.

Introduce the notation: $AB = a$, $BC = b$, $AE = p_1$, $AF = p$, and $b^2 - a^2 = m^2$. Then the segment p_1 can be represented as

$$p_1 = \left(b \sin\theta_1 \pm \sqrt{b^2 \sin^2\theta_1 - m^2}\right) \cos\theta_1. \tag{1}$$

The sign of the second term of expression (1) is chosen depending on the assembly conditions of the mechanism.

The equation of the family of straight lines $u—u$ will be $x\cos\theta + y\sin\theta = p$, or

$$x\cos(\theta_1 + \gamma) + y\sin(\theta_1 + \gamma) = p_1 \cos\gamma + \sqrt{a^2 - p_1^2}\sin\gamma. \tag{2}$$

Determine the derivative of equation (2) with respect to the angle θ_1:

$$y\cos(\theta_1 + \gamma) - x\sin(\theta_1 + \gamma) = \frac{\partial p_1}{\partial\theta_1}\cos\gamma - \frac{p_1}{\sqrt{a^2 - p_1^2}}\frac{\partial p_1}{\partial\theta_1}\sin\gamma. \tag{3}$$

The partial derivative $\partial p_1/\partial\theta_1 = \Psi$ is equal to

$$\Psi = R + T \tag{4}$$

where $R = b\cos 2\theta_1$ and

Fig. 1

Figure 1: Fig. 1

$$T = \frac{(b^2 \cos 2\theta_1 + m^2) \sin \theta_1}{\sqrt{b^2 \sin^2 \theta_1 - m^2}}.$$

From equations (2) and (3), taking equation (4) into account, we obtain the parametric equations for the linearly enveloping coupler curve of the slider-crank mechanism (Fig. 1) in the form $x = x(\theta_1)$ and $y = y(\theta_1)$:

$$\begin{aligned} x &= [p_1 \cos(\theta_1 + \gamma) - \Psi \sin(\theta_1 + \gamma)] \cos \gamma + \\ &\quad + \left[\sqrt{a^2 - p_1^2} \cos(\theta_1 + \gamma) + \frac{p_1 \Psi}{\sqrt{a^2 - p_1^2}} \sin(\theta_1 + \gamma) \right] \sin \gamma, \\ y &= [p_1 \sin(\theta_1 + \gamma) + \Psi \cos(\theta_1 + \gamma)] \cos \gamma + \\ &\quad + \left[\sqrt{a^2 - p_1^2} \sin(\theta_1 + \gamma) - \frac{p_1 \Psi}{\sqrt{a^2 - p_1^2}} \cos(\theta_1 + \gamma) \right] \sin \gamma. \end{aligned} \tag{5}$$

The current coordinates x_M, y_M of the curve linearly enveloping the straight line u—u are easily determined graphically (Fig. 1). To do this, from the instantaneous center of rotation P of the connecting rod BC, one must drop the perpendicular PM to the straight line u—u. The coordinates $x_{M'}, y_{M'}$ of the curve linearly enveloping the connecting rod BC are determined if the perpendicular PM' is dropped to the direction BC.

Fig. 1

The angle α of rotation of the link AB is related to the angle θ_1 by the condition

$$\cos \theta_1 = \frac{a}{b} \sin \alpha \tag{6}$$

The parametric equations of the curve linearly enveloping the connecting rod BC will have the form

$$\begin{aligned} x &= p_1 \cos \theta_1 - \Psi \sin \theta_1, \\ y &= p_1 \sin \theta_1 + \Psi \cos \theta_1, \end{aligned} \tag{7}$$

since for the straight line BC the angle $\gamma = 0$.

If the angle $\gamma = 90^\circ$, then the straight line u—u passes into the straight line u'—u' (Fig. 1). The coordinates $x_{M''}, y_{M''}$ of the curve linearly enveloping the straight line u'—u' will be found if from the point P the perpendicular PM''

is dropped to the straight line u'—u'. The parametric equations for this curve will be

$$x = -\sqrt{a^2 - p_1^2} \sin\theta_1 + \frac{p_1 \Psi}{\sqrt{a^2 - p_1^2}} \cos\theta_1,$$

$$y = \sqrt{a^2 - p_1^2} \cos\theta_1 + \frac{p_1 \Psi}{\sqrt{a^2 - p_1^2}} \sin\theta_1. \tag{8}$$

If the lengths of the links of the mechanism satisfy the condition $a = b$, then the motion of the link BC will have the character of Cardano motion, in which any straight line belonging to the plane of the connecting rod BC will have, as a linearly enveloping curve, a straight line or a deformed astroid, with equations for the case $\gamma = 0$

$$x = 2a \sin^3\theta_1, \qquad y = 2a \cos^3\theta_1. \tag{9}$$

The loci of points E and F (Fig. 1) will be the pole curves of the curves linearly enveloping the straight lines u—u and BC, if point A is chosen as the pole of the pole curves. The equations of the pole curve formed by point E will be

$$x = p_1 \cos\theta_1, \qquad y = p_1 \sin\theta_1, \tag{10}$$

and, respectively, for point F,

$$x = -\left(p_1 \cos\gamma + \sqrt{a^2 - p_1^2} \sin\gamma\right) \cos(\theta_1 + \gamma);$$

$$y = \left(p_1 \cos\gamma + \sqrt{a^2 - p_1^2} \sin\gamma\right) \sin(\theta_1 + \gamma), \tag{11}$$

where p_1 is determined by equation (1).

For the case when $a = b$, the equations of the pedals generated by the points E and F will have, in polar form, the form

$$p_1 = a \sin 2\theta_1; \tag{12}$$

$$p_1 = a \sin(2\theta_1 + \gamma). \tag{13}$$

In this case, for any value of the angle γ, the equations of the pedals generated by the points E and F will be four-petaled roses.

Fig. 2

Figure 2: Fig. 2

If, in a four-link mechanism with one translational pair, the driven and driving links enter into revolute pairs with the frame, then we obtain the slider mechanism ABC, shown in Fig. 2, for which it is necessary to consider line-enveloping coupler curves for straight lines rigidly connected with the slider.

Choose in the plane of the slider an arbitrary straight line u—u, passing at an angle γ to the direction BC of the axis of sliding of the slider. From the point A drop the perpendiculars AF and AE to the straight lines u—u and BC. Introduce the notation $AB = a$, $AC = b$, $AE = p_1$ and $AE = p$. The segment p_1 will be equal to

$$p_1 = b\cos\theta_1. \tag{14}$$

Fig. 2

The equation of the family of straight lines will be $x\cos\theta + y\sin\theta = p$, or

$$x\cos(\theta_1+\gamma) + y\sin(\theta_1+\gamma) = b\cos\theta\cos\gamma + \sqrt{a^2 - b^2\cos^2\theta_1}\,\sin\gamma. \tag{15}$$

The derivative with respect to the angle θ_1 of equation (15) will be

$$y\cos(\theta_1+\gamma) - x\sin(\theta_1+\gamma) = \frac{b^2\sin\theta_1\cos\theta_1}{\sqrt{a^2 - b^2\cos^2\theta_1}}\sin\gamma - b\sin\theta_1\cos\gamma. \tag{16}$$

From equations (15) and (16) we obtain parametric equations for the line-enveloping curve of the slider mechanism (Fig. 2) in the form $x = x(\theta_1)$ and $y = y(\theta_1)$:

$$x = b\cos^2\gamma + \left[\sqrt{a^2 - b^2\cos^2\theta_1}\cos(\theta_1+\gamma) - \frac{b^2\sin\theta_1\cos\theta_1}{\sqrt{a^2 - b^2\sin^2\theta_1}}\sin(\theta_1+\gamma)\right]\sin\gamma,$$

$$\tag{17}$$

$$y = b\sin\gamma\cos\gamma + \left[\sqrt{a^2 - b^2\cos^2\theta_1}\sin(\theta_1+\gamma) + \frac{b^2\sin\theta_1\cos\theta_1}{\sqrt{a^2 - b^2\sin^2\theta_1}}\cos(\theta_1+\gamma)\right]\sin\gamma.$$

The current coordinates x_M, y_M of the curve line-enveloping the straight line u—u are easily determined graphically if, from the instantaneous center of rotation P of the slider, the perpendicular PM is dropped to the straight line u—u.

The angle α of rotation of the link AB is related to the angle θ_1 by the condition

$$\operatorname{tg}\theta_1 = \frac{b - a\cos\alpha}{a\sin\alpha}. \tag{18}$$

Equations (17) will be the equations of an ellipse or a hyperbola. The latter is not difficult to verify if one takes the angle $\gamma = 90^\circ$ and from equations (17)

exclude the angle θ_1. If $b > a$, then the linearly enveloping curve will be the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2 - a^2} = 1, \tag{19}$$

and if $b < a$—the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - b^2} = 1. \tag{20}$$

As was shown earlier ([2]), the mechanism ABC (Fig. 2) can always be replaced by the equivalent mechanism $A_0B_0C_0$. The point A_0 is the foot of the perpendicular dropped from the point A to the straight line passing through the point C (C_0) at the angle $90-\gamma$. The point B_0 is the foot of the perpendicular C_0B_0, dropped from the point C (C_0) to the straight line $u-u$. The straight line $u-u$, if it is regarded as belonging to the mechanism $A_0B_0C_0$, will envelop the same ellipse or hyperbola as in the case when it belongs to the mechanism ABC. The point A_0 is the center of the ellipse or hyperbola, and the point C (C_0) is one of the foci. Thus, the linearly enveloping curves of the slider mechanism will always be ellipses or hyperbolas. If the angle $\gamma = 0$, then the point C will be the envelope. If the lengths of the links satisfy the condition $b = a$, then the envelope will also be a point, but lying to the left of the point A at the distance a.

The equation of the pedal of the curve linearly enveloping the straight line $u-u$, if the point A is chosen as the pole of the pedal, will have, in polar form, the form

$$p = b\cos(\theta - \gamma)\cos\gamma + \sqrt{a^2 - b^2\cos^2(\theta - \gamma)}\sin\gamma. \tag{21}$$

If the angle $\gamma = 90^\circ$, then equation (21) takes the form

$$p = \sqrt{a^2 - b^2\cos^2\theta}. \tag{22}$$

Passing to a rectangular coordinate system, we obtain

$$(x^2 + y^2)^2 = a^2x^2 \pm b^2y^2. \tag{23}$$

This is a curve of the 4th order, the so-called pedal curve of the type of Vitkov's curves ([2]). The upper sign in the equation will correspond to the pedal curve of an ellipse, and the lower sign to the pedal curve of a hyperbola.

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CITED LITERATURE

[1] I. I. Artobolevskii, DAN, **132**, No. 1 (1960). [2] I. I. Artobolevskii, *Theory of Mechanisms for Reproducing Plane Curves*, Publishing House of the Academy of Sciences of the USSR, 1959.

Note: Figure translations are in progress. See original paper for figures.