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Abstract

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MATHEMATICS

R. G. MAMEDOV

APPROXIMATION OF FUNCTIONS BY GENERALIZED LINEAR LANDAU OPERATORS

(Presented by Academician V. I. Smirnov on 24 II 1961)

1. Let $f(t)$ be defined and integrable on $[0, 1]$. Put

$$L_n(f; x) = L_n[f(t); x] = \frac{kn^{1/2k}}{\Gamma(1/2k)} \int_0^1 f(t) [1 - (t-x)^{2k}]^n dt, \quad (1)$$

where k is any positive integer.

It is obvious that the linear operator (1) transforms every function $f(t)$ into an algebraic polynomial of degree $\leq 2kn$. For $k = 1$, the linear operator (1) becomes the classical singular Landau integral, which is used in the proof of the Weierstrass theorem (see, for example, (1, 2)).

In this note we study the approximation of functions by the linear operators (1) on $(0, 1)$, and on the basis of (1) construct other linear operations approximating, on the infinite intervals $(0, \infty)$ and $(-\infty, \infty)$, a given function.

2. Let $0 < a, b < 1$ and $0 < q < 1$ be arbitrary numbers. Then the following assertions are valid:

Theorem 1. If $f(x)$ is continuous on $[0, 1]$, then

$$\lim_{n \rightarrow \infty} L_n(f; x) = f(x) \quad (2)$$

uniformly for any $a \leq x \leq b$.

Theorem 2. If $f(x)$ is defined on $[0, 1]$ and $f(x) \in L([0, 1])$, then the limiting equality (2) is valid at all points of continuity of the function $f(x)$ in $(0, 1)$.

Theorem 3. If $f(x)$ is continuous on $[0, 1]$, then

$$|L_n(f; x) - f(x)| \leq \left[\frac{\Gamma(1/k)}{\Gamma(1/2k)} + 1 \right] \omega_f\left(\frac{1}{n^{1/2k}}\right) + O\left(\frac{1}{n^q}\right)$$

uniformly on $[a, b]$, where $\omega_f(\delta)$ is the modulus of continuity of the function $f(x)$.

Theorem 4. If at the point $t = x \in (0, 1)$ the function $f(t)$ has a second finite derivative $f''(x)$, then the equality

$$\lim_{n \rightarrow \infty} n^{1/k} [L_n(f; x) - f(x)] = \frac{f''(x)\Gamma(3/2k)}{2\Gamma(1/2k)} \quad (k > 1),$$

is valid, i.e.

$$L_n(f; x) - f(x) = \frac{f''(x)\Gamma(3/2k)}{2\Gamma(1/2k)} \frac{1}{n^{1/k}} + o\left(\frac{1}{n^{1/k}}\right).$$

Theorem 5. Let $f(x)$ be differentiable on $[0, 1]$ N times. If $f^{(N)}(x)$ is continuous on $[0, 1]$, then

$$\lim_{n \rightarrow \infty} \frac{d^N L_n(f; x)}{dx^N} = f^{(N)}(x) \quad (3)$$

uniformly on any $[a, b] \subset (0, 1)$.

Theorem 6. If $f(x) \in L^p([0, 1])$ ($p \geq 1$), then

$$\|L_n(f; x) - f(x)\|_p = \left\{ \int_0^1 |L_n(f; x) - f(x)|^p dx \right\}^{1/p} \rightarrow 0$$

as $n \rightarrow \infty$.

The proofs of Theorems 1-5 are based on the following lemmas.

Lemma 1. Uniformly on $[a, b]$ the equality

$$L_n(1; x) = 1 + O\left(\frac{1}{n^q}\right)$$

holds.

Lemma 2. Uniformly on $[a, b]$ the equality

$$L_n[(t-x)^2; x] = \frac{\Gamma(3/2k)}{\Gamma(1/2k)} \frac{1}{n^{1/k}} + O\left(\frac{1}{n^{q+1/k}}\right)$$

holds.

Lemma 3. Uniformly on $[a, b]$ the equality

$$L_n(|t - x|; x) = \frac{\Gamma(1/k)}{\Gamma(1/2k)} \frac{1}{n^{1/2k}} + O\left(\frac{1}{n^{q+1/k}}\right)$$

holds.

The validity of some of these theorems also follows from the general theorems of P. P. Korovkin ⁽²⁾ and from the theorems contained in ^(3,4).

Theorem 6 follows from a general theorem of Orlicz ⁽⁵⁾.

3. Obviously, the operations (1) approximate functions on a finite interval. However, they can be modified so that they serve as approximating means for the approximation of functions on infinite intervals.

Theorem 7. If the function $f(x)$ is defined and bounded on $(0, \infty)$ and is such that there exists

$$P_n(f; x) = \frac{kn^{1/2k}}{\Gamma(1/2k)} \int_0^1 f(n^\alpha t) \left[1 - \left(t - \frac{x}{n^\alpha}\right)^{2k}\right]^n dt \quad (4)$$

$$(k, n = 1, 2, \dots),$$

where α is a fixed positive number, $0 < \alpha < 1/2k$, then

$$\lim_{n \rightarrow \infty} P_n(f; x) = f(x) \quad (5)$$

at every point $t = x \in (0, \infty)$ of continuity of the function $f(x)$.

Moreover, if $f(x)$ is continuous on $(0, \infty)$, then the limiting equality (5) holds uniformly on any interval $[c, d]$ contained inside the interval $(0, \infty)$.

Theorem 8. If the function $f(x)$ is defined and bounded on $(-\infty, \infty)$ and, moreover, there exists

$$P_n^*(f; x) = \frac{kn^{1/2k}}{\Gamma(1/2k)} \int_{-1}^1 f(n^\alpha t) \left[1 - \left(t - \frac{x}{n^\alpha}\right)^{2k}\right]^n dt \quad (6)$$

$$(k, n = 1, 2, \dots),$$

where $0 < \alpha < 1/2k$, then

$$\lim_{n \rightarrow \infty} P_n^*(f; x) = f(x) \quad (7)$$

at every point $t = x \in (-\infty, \infty)$ of continuity of the function $f(x)$.

Moreover, if $f(x)$ is continuous on $(-\infty, \infty)$, then equality (7) holds uniformly on every interval $[c, d]$ contained inside the interval $(-\infty, \infty)$.

For $k = 1$, Theorem 8 implies a theorem of L. C. Hsu ⁽⁶⁾.

4. Similarly, for the approximation of functions of several variables on finite and infinite domains, linear operations are constructed. For example, in the space of m dimensions the linear operators (1) and (6) are taken, respectively, in the following form:

$$L_n(f; x_1, \dots, x_m) = \left[\frac{kn^{1/2k}}{\Gamma(1/2k)} \right]^m \int_0^1 \dots \int_0^1 f(t_1, \dots, t_m) \times \\ \times \prod_{j=1}^m [1 - (t_j - x_j)^{2k}]^n dt_1 \dots dt_m;$$

$$P_n^*(f; x_1, \dots, x_m) = \left[\frac{kn^{1/2k}}{\Gamma(1/2k)} \right]^m \int_0^1 \dots \int_0^1 f(n^\alpha t_1, \dots, n^\alpha t_m) \times \\ \times \prod_{j=1}^m \left[1 - \left(t_j - \frac{x_j}{n^\alpha} \right)^{2k} \right]^n dt_1 \dots dt_m,$$

where $0 < \alpha < 1/2k$.

Assertions analogous to Theorems 1-5 also hold for the approximation of functions on the interval $[0, 1]$ by the following summatory analogue of operator (1):

$$L_n(f; x) = \frac{k}{n^{\frac{2k-1}{2k}} \Gamma(1/2k)} \sum_{j=0}^n f\left(\frac{j}{n}\right) \left[1 - \left(\frac{j}{n} - x\right)^{2k} \right]^n.$$

For the approximation of functions in $L^p([0, 1])$, the corresponding linear operator is constructed.

Theorems 7 and 8 are valid, respectively, for the summatory analogues of operators (4) and (6):

$$Q_n(f; x) = \frac{k}{n^{\frac{2k-1}{2k}} \Gamma(1/2k)} \sum_{j=0}^n f\left(\frac{j}{n^\alpha}\right) \left[1 - \left(\frac{j}{n} - \frac{x}{n^\beta}\right)^{2k} \right]^n;$$

$$Q_n^*(f; x) = \frac{k}{n^{\frac{2k-1}{2k}} \Gamma(1/2k)} \sum_{j=-n}^n f\left(\frac{j}{n^\alpha}\right) \left[1 - \left(\frac{j}{n} - \frac{x}{n^\beta}\right)^{2k} \right]^n;$$

$$(\alpha + \beta = 1, \alpha > 0, \beta > 0).$$

Let us note that the indicated linear operations, generally speaking, give a worse order of approximation of functions in comparison with some known linear operations. However, they make it possible to construct linear operations approximating a given function on any finite or infinite interval.

Institute of Mathematics and Mechanics
Academy of Sciences of the Azerbaijan SSR

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Note: Figure translations are in progress. See original paper for figures.

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