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Abstract

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ON THE CAUCHY PROBLEM FOR A CLASS OF NONLINEAR PARABOLIC SYSTEMS

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In the work (^{1}), one of the authors defined a new class of linear parabolic systems with various highest orders of differentiation with respect to the spatial coordinates (2**b**-parabolic systems), constructed the fundamental matrix of solutions of such systems, with the aid of which theorems on the correct solvability of the Cauchy problem in classes of rapidly increasing functions for linear systems and systems close to linear ones are easily established, by means of the method developed in S. D. Eidelman' s dissertation (^{2}). In the present note we set forth results on the Cauchy problem for nonlinear equations with various highest orders of differentiation, generalizing the results of (^{2}). For the strongly parabolic systems defined below, certain theorems are established on the local solvability of the Cauchy problem under smoothness assumptions on the right-hand sides of the system that are weaker than in the general case.

1. Consider the system of differential equations

$$\frac{\partial^{n_i} u_i}{\partial t^{n_i}} = F_i\left(t, x, u_1, \dots, u_N, \dots, \frac{\partial^{k_0}}{\partial t^{k_0}} D^k u_j, \dots\right) \quad (i = 1, 2, \dots, N), \tag{1}$$

where $F_i\left(t, x, u_1, \dots, u_N, \dots, \frac{\partial^{k_0}}{\partial t^{k_0}} D^k u_j, \dots\right)$ are functions of $u_1, \dots, u_N$ and their derivatives of orders $k_0 + |k|$, $k_0 + \tilde{k} \leq n_j$ with respect to t, x. Here

$$D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}},$$

$$|k| = k_1 + k_2 + \dots + k_n; \qquad \tilde{k} = \frac{k_1}{2b_1} + \frac{k_2}{2b_2} + \dots + \frac{k_n}{2b_n}; \qquad \mathbf{2b} = (2b_1, 2b_2, \dots, 2b_n),$$

$$b_1 \geq b_2 \geq \dots \geq b_n; \qquad \sigma^k = \sigma_1^{k_1} \sigma_2^{k_2} \cdots \sigma_n^{k_n}; \qquad x = (x_1, \dots, x_n); \qquad \Delta = 0 \text{ when } \frac{b_1}{b_n} \text{ is an integer,}$$

$$\Delta = 1 \text{ when } \frac{b_1}{b_n} \text{ is not an integer.}$$

Definition. System (1) is called 2**b-parabolic** in the domain

$$\Pi_1 \left\{ 0 \le t \le T;\ -\infty < x_s < \infty,\ s = 1, 2, \dots, n;\ \left| \frac{\partial^{k_0}}{\partial t^{k_0}} D^k u_j \right| \le M,\ j = 1, 2, \dots, N,\ k_0 + \tilde{k} \le n_j \right\},$$

if the roots of the equation

$$D(\lambda, \sigma) = \det \left\{ \left\| \sum_{k_0 + \tilde{k} = n_j} \frac{\partial F_i}{\partial \left[\dfrac{\partial^{k_0}}{\partial t^{k_0}} D^k u_j \right]} (i\sigma)^k \lambda^{k_0} \right\| - \left\| \begin{matrix} \lambda^{n_1} \\ \vdots \\ \lambda^{n_N} \end{matrix} \right\| \right\} = 0$$

satisfy the inequality $\operatorname{Re} \lambda < -\delta$ for

$$\left(t, x, \frac{\partial^{k_0}}{\partial t^{k_0}} D^k u_i \right) \in \Pi_1$$

and for all real $\sigma_1, \sigma_2, \dots, \sigma_n$ satisfying the condition

$$\sigma_1^{2b_1} + \sigma_2^{2b_2} + \dots + \sigma_n^{2b_n} = 1.$$

For system (1) one considers the Cauchy problem

$$\left. \frac{\partial^{k_0}}{\partial t^{k_0}} u_i \right|_{t=-t_0} = \varphi_i^{(k_0)}(x) \quad (k_0 = 0, 1, \dots, n_i - 1;\ i = 1, 2, \dots, N). \tag{2}$$

For simplicity we shall formulate all results for system (1) in which $n_1 = n_2 = \dots = n_N = 1$.

Theorem 1. If $F_i(t, x, y_1, \dots, y_\nu)$ are continuous in Π_1 with respect to t and have there continuous bounded derivatives with respect to $x_1, y_1, \dots, y_\nu$ up to order

$$r_1 = 2b_1 + 2 \left[\frac{b_1}{b_n} \right] + 1 + \Delta,$$

satisfying the Lipschitz condition with respect to $y_1, y_2, \dots, y_\nu$ in Π_1 (the derivatives of

$$\frac{\partial F_i}{\partial [D^k u_j]}, \quad \tilde{k} = 1,$$

satisfy the Lipschitz condition, in addition to $y_1, y_2, \dots, y_\nu$, also with respect to x in Π_1); moreover, the continuity with respect to t of

$$\frac{\partial F_i}{\partial [D^k u_j]}, \quad \tilde{k} = 1,$$

is uniform in $x, y_1, y_2, \dots, y_\nu$ from the domain Π_1; and $\varphi_i(x)$ have continuous bounded derivatives of order

$$|k| + \left[\frac{b_1}{b_n}\right] + 1, \quad \tilde{k} \le 2 + \frac{1}{2b_n},$$

then for $t_0 < t \le t_0 + \eta,\ \eta > 0$, there exists a solution $u_i(x,t)$ of problem (1), (2), having continuous bounded derivatives with respect to x of order

$$|k| + \left[\frac{b_1}{b_n}\right] + 2, \quad \tilde{k} \le 2 - \frac{1}{b_1}.$$

This solution is unique in the class of functions having bounded derivatives with respect to x, continuous in the Hölder sense, of order

$$|k| + \left[\frac{b_1}{b_n}\right] + 1, \quad \tilde{k} \le 2 - \frac{1}{b_1}.$$

The solution obtained depends continuously on the initial data in the following sense: let $\tilde{u}_i(x,t)$, $\tilde{\tilde{u}}_i(x,t)$ be solutions of system (1) constructed from the initial data $\widetilde{\varphi}_i(x)$, $\widetilde{\widetilde{\varphi}}_i(x)$, respectively, and suppose

$$\sum_{i=1}^{N} \sum_{\tilde{k}<1,\ |s| \le \left[\frac{b_1}{b_n}\right]+1} \left| D^{k_i+s} \left[\widetilde{\varphi}_i(x) - \widetilde{\widetilde{\varphi}}_i(x)\right] \right| < \varepsilon;$$

then

$$\left| \tilde{u}_i(x,t) - \tilde{\tilde{u}}_i(x,t) \right| \le M_1 \varepsilon,$$

where M_1 depends on M, T, δ (δ from the parabolicity condition) and the constants bounding the functions F_i and their derivatives.

If, however,

$$r_1 = 2b_1 + 2\left[\frac{b_1}{b_n}\right] + 1 + \Delta + S,$$

then every solution of the system that has bounded derivatives with respect to x, continuous in the Hölder sense, of order

$$|k| + \left[\frac{b_1}{b_n}\right] + 1, \quad \tilde{k} \leq 2 - \frac{1}{b_1},$$

has continuous bounded derivatives of order

$$|k| + \left[\frac{b_1}{b_n}\right] + 1, \quad \tilde{k} \leq 2 + \frac{1}{2b_n} + \frac{S}{2b_1}, \quad \text{for } t > t_0.$$

The proof of the theorem is carried out by reducing the problem under consideration to an equivalent Cauchy problem for a special quasilinear parabolic system of the form

$$\frac{\partial u_i}{\partial t} = \sum_{j=1}^{N_1} \sum_{\widetilde{m}=1} A_{ij}^{(m)}(t, x, u_1, \dots, u_{N_1}, \dots, D^k u_j, \dots)\, D^m u_j +$$

$$+ F_{1i}(t, x, u_1, \dots, u_{N_1}, \dots, D^{k'} u_j, \dots), \quad i = 1, 2, \dots, N_1, \tag{3}$$

where

$$\tilde{k} \leq \frac{l}{2b_n}, \quad l < 2b_n - 1; \quad \widetilde{k'} < 1,$$

with initial conditions

$$u_i|_{t=-t_0} = \psi_i(x), \quad i = 1, 2, \dots, N_1. \tag{4}$$

2. For the quasilinear system (3) the following theorem holds.

Theorem 2. If:

1) the coefficients $A_{ij}^{(m)}(t, x, y_1, \dots, y_{\nu_1})$ are defined in the domain $Q_{\nu_1}\{t_0 \leqslant t \leqslant T,\ -\infty < x_s < \infty,\ s = 1, 2, \dots, n,\ |y_j| \leqslant M,\ j = 1, 2, \dots, \nu_1\}$ and are continuous in it with respect to t, uniformly with respect to $x, y_1, y_2, \dots, y_{\nu_1}$ from Q_{ν_1};
   $F_{1i}(t, x, y_1, \dots, y_{\nu_2})$ are defined in Q_{ν_2} and are continuous in it with respect to t;

2) $A_{ij}^{(m)}(t, x, y_1, \dots, y_{\nu_1})$ and $F_{1i}(t, x, y_1, \dots, y_{\nu_2})$ have, respectively in Q_{ν_1} and Q_{ν_2}, $r_1 = r + p + 2$, $r = \left[\frac{lb_1}{b_n}\right]$, $p \geqslant 0$, continuous bounded derivatives with respect to $x, y_1, \dots, y_{\nu_1}$ $(y_1, \dots, y_{\nu_2})$, satisfying the Lipschitz condition with respect to $y_1, y_2, \dots, y_{\nu_1}$ $(y_1, y_2, \dots, y_{\nu_2})$ in Q_{ν_1} (Q_{ν_2}), and the derivatives $A_{ij}^{(m)}(t, x, y_1, y_2, \dots, y_{\nu_1})$, moreover, also satisfy the Lipschitz condition with respect to x;

3) $\varphi_i(x)$ have continuous bounded derivatives of order $|k|$,

$$\tilde{k} \leqslant 1 + \frac{p + r + 2}{2b_1},$$

then problem (3), (4) has a solution, bounded and continuous together with its derivatives of order $|k|$,

$$\tilde{k} \leqslant 1 + \frac{r + p}{2b_1}.$$

If

$$p = 2b_1 - r - 2 + \left[\frac{b_1}{b_n}\right] + \Delta,$$

then the solution $u_i(x, t)$ has continuous bounded derivatives of order $|k| + 1$,

$$\tilde{k} \leqslant 2 - \frac{1}{b_1}.$$

This solution is unique in the class of functions having bounded and Hölder-continuous, with respect to x, derivatives of order $|k|$,

$$\tilde{k} \leqslant 2 - \frac{1}{b_1}.$$

Under conditions 1) and 2), with

$$p \geqslant 2b_1 - r - 2 + \left[\frac{b_1}{b_n}\right],$$

every solution $u_i(x, t)$ of system (3), having bounded and Hölder-continuous, with respect to x, derivatives of order $|k|$,

$$\tilde{k} \leqslant 2 - \frac{1}{b_1},$$

in the strip $[t_0, T]$, has, for $t > t_0$, derivatives of order $|k|$,

$$\tilde{k} \leqslant 1 + \frac{r + p}{2b_1},$$

bounded on each segment $[t_0 + \delta_1, T]$, $\delta_1 > 0$.

The theorem is established by the method of successive approximations with the aid of fundamental matrices of solutions constructed in (1).

3. The stringent restrictions imposed on the right-hand sides in Theorems 1 and 2 are due to the fact that, in proving Theorem 2, one has to establish uniform-in-x continuity in t of the successive approximations, and, in proving the equivalence of problems for nonlinear and quasilinear systems, to use the uniqueness theorem for the solution of the Cauchy problem for linear systems. These restrictions can be weakened for certain strongly parabolic systems, since for them it is not necessary to require uniform continuity in t of the coefficients of the system.

Definition. System (1) $(n_1 = n_2 = \dots = n_N = 1)$ is called **strongly parabolic** in Π_1 if

$$\operatorname{Re}\left\{\sum_{i,j=1}^{N}\sum_{\tilde{k}=1}\frac{\partial F_i}{\partial[D^k u_j]}(i\sigma)^k a_i a_j\right\} < -\delta|a|^2(\sigma_1^{2b_1} + \dots + \sigma_n^{2b_n})$$

for $(t, x, D^k u) \in \Pi_1$, arbitrary real $\sigma_1, \sigma_2, \dots, \sigma_n$, and for every complex-valued vector $a = (a_1, a_2, \dots, a_n)$.

For a strongly parabolic system (3) one can establish the existence of a solution of the Cauchy problem (4) under the following assumptions:

1) The coefficients $A_{ij}^{(m)}(t, x, y_1, \dots, y_{\nu_1})$ and $F_{1i}(t, x, y_1, \dots, y_{\nu_2})$ are continuous in t in Q_{ν_1} (Q_{ν_2}) and have there 2 continuous derivatives with respect to $x, y_1, \dots, y_{\nu_1}$ $(y_1, \dots, y_{\nu_2})$, satisfying the Lipschitz condition with respect to

$y_1, y_2, \dots, y_\nu$, $u_1, y_2, \dots, y_\nu$, and the derivatives $A_{ij}^{(m)}(t, x, y_1, \dots, y_\nu)$; in addition, the Lipschitz condition in x;

2) $\psi_i(x)$ have continuous bounded derivatives of order $|k| + 2$, $\tilde{k} \leqslant 1$.

Then the solution $u_i(x, t)$ will have continuous bounded derivatives of order $|k|$, $\tilde{k} \leqslant 1$.

In the case of one spatial coordinate (in this case system (1) becomes an ordinary parabolic system in the sense of I. G. Petrovsky), the proof of the equivalence of the problems for nonlinear and quasilinear systems can be carried out without using the uniqueness theorem for the solution of the Cauchy problem. In this case the following theorem is valid.

Theorem 3. If $F(t, x, y_1, \dots, y_\nu)$ has 4 continuous and bounded derivatives with respect to $x, y_1, \dots, y_\nu$, satisfying the Lipschitz condition with respect to $y_1, \dots, y_\nu$, in the domain Π_1; $\varphi(x)$ has $2b{+}4$ continuous bounded derivatives, then for $t_0 < t \leqslant t_0 + \eta$ there exists a solution of the Cauchy problem $u|_{t=+t_0} = \varphi(x)$ for system (1), differentiable $2b + 2$ times.

4. The fact that the smoothness of solutions of the Cauchy problem increases when the smoothness of the functions F_i is increased, established in Theorem 1, makes it possible to prove a number of assertions on the continuation of solutions possessing a certain special property (property II) up to the boundary of the domain in which the sufficiently smooth functions F_i are defined. The proof of the continuation theorems is carried out by means of the method of M. A. Krasnosel' skii and S. G. Krein (^{3}).

To study the continuability of solutions $u_i(x,t)$ having bounded derivatives of order $|k|$, $\tilde{k} \leqslant 1$, with respect to x for $-\infty < x_s < \infty$, $s = 1, 2, \dots, n$, one assigns to each solution a "trajectory"

$$y(t) = \{y_j(t)\}_{j=1}^{\nu}, \qquad y_j(t) = \sup_x |D^k u(x,t)|, \quad \tilde{k} \leqslant 1, \quad j = 1, 2, \dots, \nu.$$

Let Q be a domain of the ν-dimensional space $y_1, y_2, \dots, y_\nu$, defined by the inequalities $y_j < M$, $j = 1, 2, \dots, \nu$. Suppose that the requirements of existence theorem 1 are fulfilled for $F_i(t, x, y_1, \dots, y_\nu)$ for $-\infty < x_s < \infty$, $s = 1, \dots, n$, $(y_1, \dots, y_\nu) \in Q$, and all t occurring in our consideration.

Property II. Let $u_i(x,t)$ be defined on a finite interval (t_0, t_1) and $y(t) \in Q_1$, $\overline{Q}_1 \subset Q$; then, whatever $\tilde{t} \in (t_0, t_1)$ we take, there exists an $\eta > 0$, independent of $\tilde{t}$, such that the solution $u_i(x,t)$ is defined on the half-segment $(t_0, \tilde{t} + \eta)$.

It can be shown, for example, that property II is possessed by smooth solutions of systems of the form

$$\frac{\partial u}{\partial t} = \sum_{\tilde{k}=1} A_k(x,t) D^k u + F(t, x, D^{k'} u),$$

$$\tilde{k}' \leqslant 1 - \frac{1}{2b_n}, \qquad b_n = \min_j b_j.$$

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