DIVERGENT DIAGRAMS IN THE FORMAL PERTURBATION THEORY FOR A NONIDEAL FERMI-DIRAC SYSTEM AND THEIR MUTUAL COMPENSATION

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DIVERGENT DIAGRAMS IN THE FORMAL PERTURBATION THEORY FOR A NON-IDEAL FERMI-DIRAC SYSTEM AND THEIR MUTUAL COMPENSATION

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The formal perturbation-theory series for the energy of the ground state of a nonideal Fermi–Dirac system, the so-called Brackner—Goldstone expansion, has been repeatedly and thoroughly investigated. However, it seems that due attention has not yet been paid to the existence in this formal expansion, in high orders, of divergent diagrams whose divergences arise from the vanishing of energy denominators on the Fermi surface and are not connected with the singular character of the interaction, which in what follows is assumed to be regular.

In the present note the results of an investigation of these divergences will be given. It will be shown that in the first high orders of perturbation theory these divergences compensate one another.

The contribution from some diagram of the n-th order of perturbation theory, or rather, from one of its time versions, can in general form be represented in the following way:

$$\int \cdots \int \frac{f(\mathbf{k}_1, \dots, \mathbf{k}_m)}{D_1 \cdots D_{n-1}}\, d\mathbf{k}_1 \cdots d\mathbf{k}_m, \tag{1}$$

where $\mathbf{k}_1, \dots, \mathbf{k}_m$ are the momenta numbering the particle and hole lines in the diagram (with account of the momentum conservation laws at the vertices of the diagram); $D_1, \dots, D_{n-1}$ are the energy denominators (in an n-th order diagram there are $n-1$ of them); $f(\mathbf{k}_1, \dots, \mathbf{k}_m)$ vanishes if any particle momentum takes, in absolute value, a value greater than the Fermi momentum k_F, or if any hole momentum takes, in absolute value, a value less than the Fermi momentum k_F. The structure of an individual energy denominator is as follows:

$$D = \sum \text{particle energies} - \sum \text{hole energies}, \tag{2}$$

and, by virtue of the law of conservation of the number of particles, the number of particle energies in the denominator is equal to the number of hole energies.

As has already been noted many times, the energy denominators inside the region of integration, by virtue of (2), are always positive and, consequently, do not vanish. They may vanish, and in fact do vanish, on the boundaries of the region of integration.

Thus, (1) is an integral with a singular integrand. Therefore the question of the convergence of this integral naturally arises.

One can formulate a criterion for the divergence of the integral from the immediate neighborhood of a boundary region in which some group of k denominators in (1) simultaneously vanishes.

Let the number of different particle and hole energies entering into a group of k denominators be equal to l. Setting a separate denominator equal to zero, by virtue of (2), is equivalent to equating modulo the Fermi momentum k_F all the particle and hole momenta entering into it. Thus, the boundary region in which all denominators of the group simultaneously vanish is determined by l one-dimensional conditions imposed on the integration variables in (1), and, consequently, the boundary region has l dimensions fewer than the main region of integration in (1).

The divergence of the integral (1) from the immediate neighborhood of the boundary region under consideration is therefore determined by the divergence of the integral

$$\int_0 dx \, \frac{x^{l-1}}{x^k}, \tag{3}$$

which diverges only if $k \geq l$. This is the formulated criterion for divergence. In the case of equality the divergence is logarithmic.

Let us show, with the help of the divergence criterion, that diagrams having no self-energy insertions cannot diverge. In such diagrams the momenta standing on the lines cannot coincide with one another and therefore are all different.

Take a group of k denominators. All of them must be different. We must show that for this group $l > k$. Consider all the lines that are crossed by the cuts defining the chosen group of k denominators. Into the group of vertices between two neighboring cuts, as well as into each of the two outer groups of vertices, enter and leave at least two of the lines under consideration. Since the number of the indicated groups is $k + 1$, consequently the total number of lines under consideration is not less than $2(k + 1) = 2k + 2$, whence the assertion to be proved follows.

The number l for any group of k denominators cannot be less than 4, for in each denominator there are at least 2 particle and 2 hole energies. Thus, for the order n of a divergent diagram we obtain the restriction

$$n - 1 \geq k \geq l \geq 4, \tag{4}$$

from which we conclude that the divergences under consideration begin with the 5th order. Diagrams of the 3rd and 4th orders do not diverge.

Thus, diagrams without self-energy insertions will not diverge. Therefore, divergent diagrams should be sought among diagrams with such insertions, i.e., among diagrams with identical denominators.

Such diagrams do indeed exist, beginning with the 5th order. These latter have the form of one of the two possible 2nd-order diagrams, in the lines of which three 1st-order self-energy insertions have been made, so that the diagrams have 4 identical denominators, between which one 1st-order self-energy insertion has been made in one of the possible 4 lines.

It is easy to see that if we add together the contributions from all the indicated divergent 5th-order diagrams, then as a result we obtain compensation of the divergences. Indeed, we can carry out a summation that excludes from consideration the 1st-order insertions. Such a summation reduces to replacing the unperturbed one-particle energies $E(k)$ by $E'(k)$, where

$$E'(k) = E(k) + \frac{1}{V} \sum_{k'} (2v(0) - v(k - k')), \tag{5}$$

$$|k'| < k_F$$

In turn, $v(k)$ represents the interaction. The indicated diagrams of fifth order are obtained in such a partially summed perturbation theory by further expanding the second-order diagrams in the interaction entering the denominator by means of (5). Under such a further expansion, however, as the power of the denominator that vanishes on the Fermi surface increases, the power of the numerator, which also vanishes on the Fermi surface, increases to the same degree. A compensation of divergences arises, as was stated above.

There is a general reason for the appearance of a compensating factor in the situation where between two identical denominators some one and the same insertion is made into only one line, or rather one of its versions (the insertion cannot be represented in the form of two parts connected with each other only by one particle or hole line).

On the Fermi surface the contribution from the insertion is determined only by the structure of the insertion itself, since the appearance in the denominators of Fermi energies from cutting the insertion and also other lines between the identical denominators on which Fermi momenta stand is easily taken into account by introducing a fictitious line closing the insertion (whether the fictitious line is a particle or a hole line must be taken into account when composing the energy denominator from the cut intersecting this line; this need not be taken into account when composing the sign factor both from the closed loop and from the hole line).

Now, between two identical denominators there is an equal number of hole and particle lines. An insertion into a particle line increases by one the number of particle lines and does not change the number of hole lines in the whole diagram, i.e., it does not lead to a change of the sign factor. The same insertion into a hole line increases by one the number of hole lines and does not change the number of particle lines, i.e., it leads to a change of the sign factor to the opposite one. Thus, the insertion into a particle line is exactly compensated by the insertion into a hole line, whence the origin of the compensating factor in the situation under consideration is directly visible.

The result obtained sharply raises the order in which one may expect the appearance of noncompensating divergent diagrams. Taking it into account, a noncompensating divergence can arise only in the fourteenth order. The divergent diagrams in this order have the form of fifth-order diagrams, only between two identical denominators two insertions are made either into two different lines or into one and the same line, when this is possible (of course, one should take into consideration only overlapping mutual time combinations of such insertions). The indicated diagrams, however, are also compensated.

Let us dwell in more detail on the occurrence of the compensating factor in a situation of the type described. It is again sufficient to carry out the consideration only on the Fermi surface, where the simple rule with a fictitious line formulated above is applicable.

Take any two insertions, or rather certain of their time versions (the insertions cannot be represented in the form of two parts connected with each other only by one particle or hole line). We close each such insertion by a fictitious line, which corresponds to inserting them into different lines, and compose the contribution from all possible overlapping mutual time combinations of such insertions. Denote the contribution by the symbol f_{11}. Now connect these insertions by a real line and after that close the resulting composite insertion by a fictitious line, which corresponds to inserting both insertions into one and the same line. The resulting composite insertions will be of two kinds: when the real line connecting these insertions is a particle line or a hole line. The contribution from all possible composite insertions of the first kind will be denoted by f_2^+, and the contribution from all possible composite

We shall denote the insertions of the second kind by f_2^-. It is obvious from the arguments presented that

$$f_{11} = f_2^+ - f_2^-. \tag{6}$$

Since the complex insertions entering into f_2^- have, in addition, one hole line in comparison with the corresponding temporary combinations entering into f_{11}, a minus sign appears before them in (6).

Suppose that between two identical denominators there are $2S$ lines (S particle lines and S hole lines). We shall make two insertions into two lines. These two lines may be particle and particle (there are $\frac{1}{2}S(S-1)$ such cases); hole and hole (there are $\frac{1}{2}S(S-1)$ such cases); particle and hole (there are S^2 such cases). For insertions in the third case an additional minus sign arises; for insertions in the first and second cases it does not arise. Thus the total contribution from two insertions into two lines will be

$$\tfrac{1}{2}S(S-1)f_{11} + \tfrac{1}{2}S(S-1)f_{11} - S^2 f_{11} = -Sf_{11}, \tag{7}$$

We shall make two insertions into one line, more precisely, a complex insertion consisting of two. Into a particle line we can make only complex insertions combined into f_2^+, for if we make a complex insertion from f_2^-, then we arrive at an anomalous diagram (in which there is both a hole and a particle line with one and the same momentum), whose contribution is zero. For an analogous reason, into a hole line one can make only complex insertions combined into f_2^-. Upon insertion into a hole line an additional minus sign arises; upon insertion into a particle line it does not arise. Thus the total contribution from two insertions into one line will be, using (6),

$$Sf_2^+ - Sf_2^- = Sf_{11}. \tag{8}$$

Combining (7) and (8), we are convinced of the presence of a compensating factor in the situation under consideration.

It is curious to consider the situation with three insertions. In this case it is convenient to introduce into consideration the following quantities: f_{111}, f_{21}^+, f_{21}^-, f_3^{++}, f_3^{--}. It is easy to see that the relations

$$f_{111} = f_{21}^+ - f_{21}^-, \tag{9}$$

$$f_{21}^+ + f_{21}^- = f_3^{++} - f_3^{--}. \tag{10}$$

exist between them.

Using (9) and (10), it is easy to establish the presence of a compensating factor also in this situation.

We have established the compensation of divergences in the first higher orders of perturbation theory. Although we do not have a general proof of this fact, nevertheless there are no grounds for supposing that uncompensated divergences exist in perturbation theory. At the same time, the Brueckner–Goldstone expansion contains formally divergent diagrams.

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Note: Figure translations are in progress. See original paper for figures.