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# MATHEMATICS

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**Abstract**

**Full Text**

MATHEMATICS

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## THE METHOD OF ALTERNATING DIRECTIONS FOR SOLVING SYSTEMS OF FINITE-DIFFERENCE EQUATIONS

*(Presented by Academician S. L. Sobolev on 29 XII 1960)*

Among the various methods for solving systems of finite-difference equations approximating an elliptic equation, the method of alternating directions <sup>(2-4)</sup> has the important property that, for a uniform grid, in order to find the solution with accuracy  $\varepsilon$ , it requires a number of arithmetic operations asymptotically smaller than other iterative methods.

For the equation  $\Delta u - k^2 u = f$  in a rectangle, in <sup>(4)</sup> an asymptotic estimate was obtained for the number of arithmetic operations,  $\sim h^{-2} \ln h \ln \varepsilon$ , originally obtained for  $\Delta u = f$  in <sup>(2, 3)</sup>. In <sup>(4)</sup>, convergence is studied in a general domain and for boundary conditions of the form  $\partial u / \partial \nu + du = 0$ . In <sup>(5)</sup>, a variant of the method of alternating directions is studied for  $\Delta^2 u + c^2 u = f$  in a rectangle, with the values of  $u$  and  $\partial^2 u / \partial \nu^2$  prescribed on the boundary. For the polyharmonic equation in a rectangle with even normal derivatives prescribed on the boundary, the method of alternating directions was investigated in <sup>(6)</sup>.

In the present note, convergence of the method of alternating directions is established for the first boundary-value problem in a rectangle for certain self-adjoint elliptic equations of order higher than the second; moreover, for an equation with separable variables, an estimate is given for the rate of convergence.

1. In the square  $D : 0 \leq x \leq 1, 0 \leq y \leq 1$ , we seek a solution of the equation

$$\begin{aligned}
 Lu = & \frac{\partial^2}{\partial x^2} \left( a(x) \frac{\partial^2 u}{\partial x^2} \right) + 2 \frac{\partial^2}{\partial x \partial y} \left( b(x, y) \frac{\partial^2 u}{\partial x \partial y} \right) + \frac{\partial^2}{\partial y^2} \left( c(y) \frac{\partial^2 u}{\partial y^2} \right) \\
 & - \frac{\partial}{\partial x} \left( d(x) \frac{\partial u}{\partial x} \right) - \frac{\partial}{\partial y} \left( e(y) \frac{\partial u}{\partial y} \right) + f(x)u + g(y)u = h(x, y),
 \end{aligned} \tag{1}$$

satisfying the boundary conditions:

$$u = 0; \quad \frac{\partial u}{\partial \nu} = 0; \tag{2}$$

$$a > 0, \quad c > 0; \quad b, d, e, f, g \geq 0; \quad b^2 - ac \leq 0; \quad (3)$$

$$a, b, c \in C^{(3)}; \quad d, e \in C^{(2)}; \quad f, g \in C^{(1)}.$$

On the grid  $D_h : x_i = ih, y_j = jh; h = 1/N, 0 \leq i \leq N, 0 \leq j \leq N$ , define  $v_{ij}$  as  $v(ih, jh)$ . In view of (2),  $u_{ij} = 0$  if  $i$  or  $j = 0, 1, N - 1, N$ , i.e.

$$u_{ij} = 0 \quad \text{on } S_h. \quad (4)$$

For points external to the square,  $u_{ij} = 0$ , and for  $i, j \in D_h \setminus S_h$ , in order to determine  $u_{ij}$  we obtain the system of algebraic equations

$$\begin{aligned} L_h u_{ij} = & \Delta_{\bar{x}x}(a_i \Delta_{xx} u_{ij}) + 2\Delta_{\bar{x}y}(b_{ij} \Delta_{xy} u_{ij}) + \Delta_{\bar{y}y}(c_j \Delta_{yy} u_{ij}) \\ & - \Delta_x(d_i \Delta_x u_{ij}) - \Delta_y(e_j \Delta_y u_{ij}) + f_i u_{ij} + g_j u_{ij} = h_{ij}, \end{aligned} \quad (5)$$

where

$$\Delta_{\bar{x}} u_i = \frac{u_i - u_{i-1}}{h}, \quad \Delta_{xu} i = \frac{u_{i+1} - u_i}{h}.$$

The existence of a solution of (4), (5) and its convergence to the solution of (1), (2) are established in (\*). It is not difficult to verify that each of the terms in (5) is a symmetric and positive operator on the class of  $u_{ij}$  satisfying (4), and  $L_h$  is positive definite.

## 2. Introducing the notation

$$L_x u_{ij} = \Delta_{\bar{x}x}(a_i \Delta_{xx} u_{ij}) - \Delta_x(d_i \Delta_x u_{ij}) + f_i u_{ij}, \quad (6)$$

$$L_y u_{ij} = \Delta_{\bar{y}y}(c_j \Delta_{yy} u_{ij}) - \Delta_y(e_j \Delta_y u_{ij}) + g_j u_{ij}, \quad (7)$$

$$2L_{xy} u_{ij} = 2\Delta_{\bar{x}y}(b_{ij} \Delta_{xy} u_{ij}), \quad (8)$$

we define, by analogy with (\*), the following iterative process:

$$(L_x + \tau_n E) u_{ij}^{(n+1/2)} = h_{ij} - (L_y + 2L_{xy} - \tau_n E) u_{ij}^{(n)}; \quad (9)$$

$$(L_y + \tau_n E) u_{ij}^{(n+1)} = L_y u_{ij}^{(n)} + \tau_n E u_{ij}^{(n+1/2)}. \quad (10)$$

Here  $E$  is the identity operator;  $\tau_n$  is a certain iteration parameter;  $u^{(n)}, u^{(n+1)}$  are successive iterates;  $u^{(n+1/2)}$  is an intermediate vector. Eliminating  $u^{(n+1/2)}$  from (9) with the aid of (10), we have:

$$[\tau_n^2 E + \tau_n(L_x + L_y) + L_{xL}y]u^{(n+1)} = h + [\tau_n^2 E - 2\tau_n L_{xy} + L_{xL}y]u^{(n)}, \quad (11)$$

and for the error

$$u - u^{(n)} = e^{(n)}; \quad Ae^{(n+1)} = Be^{(n)}, \quad (12)$$

where

$$Ae^{(n+1)} = [\tau_n^2 E + \tau_n(L_x + L_y) + L_{xL}y]e^{(n+1)};$$

$$Be^{(n)} = [\tau_n^2 E - 2\tau_n L_{xy} + L_{xL}y]e^{(n)}.$$

Of course,  $e^{(n)} = 0$  on  $S_h$ .

**Lemma 1.** The operators  $A$  and  $B$  are symmetric, and  $(B\psi, \psi) > 0$  for  $\|\psi\| > 0$ , where

$$(\psi, \varphi) = \sum_{i=0}^N \sum_{j=0}^N \psi_{ij} \varphi_{ij} h^2.$$

**Proof.** For the symmetry of  $A$  and  $B$ , it is enough to obtain

$$L_{xL}y = L_yLx, \quad (13)$$

i.e., the commutativity of operators of the form

$$L_{xu_{ij}} = \sum_{k=-p}^p u(x_i + kh, y_j) \alpha(k, 0)$$

and

$$L_{yu_{ij}} = \sum_{l=-q}^q u(x_i, y_j + lh) \beta(0, l),$$

which, obviously, follows from the possibility of changing the order of summation and of taking outside the summation sign coefficients that do not depend on the summation index. From (13) it follows that  $L_{xL}y > 0$ .

Applying the “summation by parts” formulas (1), we obtain for  $(B\psi, \psi)$ :

$$\begin{aligned}
 (B\psi, \psi) &= \tau_n^2(\psi, \psi) - 2\tau_n(L_{xy}\psi, \psi) + (L_{xL}y\psi, \psi) \geq \\
 &\geq \tau_n^2(\psi, \psi) - 2\tau_n h^2 \max_{ij} (b_{ij}) \sum_{i,j=0}^{N-1} (\Delta_{xy}\psi_{ij})^2 + h^2 \min_{ij} (a_i c_j) \sum_{i,j=0}^{N-2} (\Delta_{xxyy}\psi_{ij})^2 \geq \\
 &\geq \tau_n^2(\psi, \psi) - 2\tau_n r (\Delta_{xxyy}\psi, \psi) + r^2 h^2 \sum_{i,j=1}^{N-1} (\Delta_{xxyy}\psi_{ij})^2 = \\
 &= h^2 \sum_{i,j=1}^{N-1} ((\tau_n r - r \Delta_{xxyy})\psi_{ij})^2 \geq 0.
 \end{aligned} \tag{14}$$

Here  $r^2 = \min_{ij} \{a_i c_j\}$ , and  $b^2 - ac \leq 0$  has been used.  $(B\psi, \psi) = 0$  if

$$\Delta_{xxyy}\psi_{ij} = \lambda\psi_{ij}$$

for  $1 \leq i, j \leq N - 1$ . Taking into account that  $\psi_{i,j} = 0$  on  $S_h$ , we arrive at the conclusion that  $\psi \equiv 0$ . The lemma is proved.

**3. Theorem 1.** For fixed  $\tau_n \geq 0$ , the iterative process (9), (10) converges in the metric

$$\|\psi\|_B^2 = (B\psi, \psi).$$

Indeed, by Lemma 1 there exists a system of functions  $\{\psi_k\}$  such that

$$(B\psi_k, \psi_l) = \delta_{kl} = \begin{cases} 0, & k \neq l, \\ 1, & k = l; \end{cases} \tag{15}$$

$$A\psi_k = \lambda_k B\psi_k. \tag{16}$$

The system  $\{\psi_k\}$  is complete in the metric  $\|\cdot\|_B$ , and consequently also in the ordinary one. Let

$$e^{(n+1)} = \sum_k \psi_k d_k^{(n+1)}, \quad e^{(n)} = \sum_k \psi_k d_k^{(n)}. \tag{17}$$

Substituting (17) into (12), we shall have:

$$Ae^{(n+1)} = \sum_k d_k^{(n+1)} A\psi_k = \sum_k d_k^{(n+1)} \lambda_k B\psi_k = \sum_k d_k^{(n)} B\psi_k. \tag{18}$$

From (18), using (15):

$$\lambda_k d_k^{(n+1)} = d_k^{(n)}. \quad (19)$$

Let us estimate  $\lambda_k$  from (16):

$$\lambda_k = (A\psi_k, \psi_k) = ((B + \tau L_h)\psi_k, \psi_k) = 1 + \tau(L_h\psi_k, \psi_k) > 1. \quad (20)$$

Therefore

$$\left| \frac{d_k^{(n+1)}}{d_k^{(n)}} \right| = \frac{1}{\lambda_k} < 1, \quad \|e^{(n+1)}\|_B \leq \max_k \frac{1}{\lambda_k} \cdot \|e^{(n)}\|_B. \quad (21)$$

Convergence will obviously also hold in the ordinary metric.

4. Let us consider separately the case  $b \equiv 0$ . In this case the following theorem is valid:

**Theorem 2.** *If  $b \equiv 0$ , then, to find the solution of the system (4), (5), it suffices, with a proper choice of  $\{\tau_n\}$ , to carry out  $\sim \ln h \ln \varepsilon$  iterations by the method of alternating directions. The number of arithmetic operations will be  $\sim h^{-2} \ln h \ln \varepsilon$ .*

**Proof.** As shown in Lemma 1,  $L_x$  and  $L_y$  commute; therefore they have a system of common eigenfunctions forming an orthonormal basis for any  $\tau_n > 0$ .

Let

$$L_x \psi_k = \lambda_k^2 \psi_k, \quad L_y \psi_k = \mu_k^2 \psi_k. \quad (22)$$

It is not difficult to obtain that  $m \leq \lambda_k^2, \mu_k^2 \leq M/h^4$ . Then

$$\left| \frac{d_k^{(n+1)}}{d_k^{(n)}} \right| = \rho_k(\lambda_k, \mu_k) = \frac{\tau_n^2 + \lambda_k^2 \mu_k^2}{\tau_n^2 + \tau_n(\lambda_k^2 + \mu_k^2) + \lambda_k^2 \mu_k^2} < 1, \quad (23)$$

where

$$\rho_k^2(\lambda_k, \mu_k) \leq \rho_k(\lambda_k, \lambda_k) \rho_k(\mu_k, \mu_k).$$

The rest of the proof proceeds as in (3), and one constructs a set  $(\tau_1, \dots, \tau_p)$  such that for each  $\psi_k$  there is a  $\tau_l$  from this set for which  $\rho_l(\lambda_k, \mu_k) \leq q < 1$ . Repeating such a cycle with  $p$  parameters  $\sim |\ln \varepsilon|$  times, we find a solution with accuracy  $\varepsilon$ , i.e.  $\|e^{(n+1)}\| \leq \varepsilon \|e^{(n)}\|$ .

The estimate of the number of arithmetic operations is obtained in the same way as in  $(^3, ^5)$ , since in the iterations only systems with pentadiagonal matrices are solved.

**Remark 1.** For the equation where  $b \equiv 0$ , one can also construct an analogue of method  $(^2)$  and obtain for it the same convergence estimate.

**Remark 2.** The results are transferred in an obvious way to certain equations of higher orders, namely those for which Lemma 1 holds.

**Remark 3.** Extensions to the case of a parallelepiped are also possible, if one considers an analogue of schemes  $(^3)$ .

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