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Abstract

Full Text

MATHEMATICS

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On Some Classes of Convex Functions

(Presented by Academician P. S. Aleksandrov on 1 X 1960)

In the first part of the monograph by M. A. Krasnosel'skii and Ya. B. Rutickii (¹), in connection with the theory of Orlicz spaces, classes of convex N -functions satisfying the so-called Δ_2 -, Δ' -, Δ_3 -, Δ^2 -, and Δ_Φ -conditions are considered (definitions of these concepts are given below). In (¹) various criteria are established for an N -function $M(u)$ to belong to one of these classes. In the present work, necessary and sufficient conditions are established for an N -function to belong to the classes under consideration, formulated in terms of properties of the complementary function.

We first recall some general facts from the theory of N -functions (see (¹)).

Let $p(t)$ ($0 \leq t < \infty$) be a nondecreasing, right-continuous function satisfying the conditions: $p(+0) = 0$, $p(t) > 0$ for $t > 0$, and $p(\infty) = \infty$. The convex function

$$M(u) = \int_0^{|u|} p(t) dt$$

is called an N -function. This function has the following properties: it is even, positive for $u \neq 0$, $M(u)/u$ is strictly increasing for $u > 0$, and

$$\frac{1}{2}up\left(\frac{1}{2}u\right) \leq M(u) \leq up(u). \quad (1)$$

Put $q(s) = \sup_{p(t) \leq s} t$. This function has the same properties as the function $p(t)$. The convex function

$$N(v) = \int_0^{|v|} q(s) ds$$

is called the N -function complementary to $M(u)$.

If the function $p(t)$ is strictly increasing and continuous, then

$$p[q(v)] = v, \quad q[p(u)] = u. \quad (2)$$

Two N -functions $M_1(u)$ and $M_2(u)$ are called equivalent (this notion of equivalence was introduced in ⁽¹⁾) if, for large values of u , the inequalities

$$M_1(k_1 u) \leq M_2(u) \leq M_1(k_2 u), \quad (3)$$

hold, where k_1, k_2 are positive numbers.

In ⁽¹⁾ it was proved that the function

$$M_1(u) = \int_0^{|u|} \frac{M(t)}{t} dt \quad (4)$$

is an N -function equivalent to $M(u)$.

Theorem. In order that an N -function $M(u)$ satisfy one of the conditions (the Δ -conditions) in the first column, it is necessary and sufficient that the complementary N -function $N(v)$ satisfy the corresponding condition (the ∇ -condition) in the second column:

for large u, v	for large u, v
$(\Delta_2) : M(2u) \leq kM(u)$	$(\nabla_2) : 2kN(u) \leq N(ku)$
$(\Delta') : M(uv) \leq kM(u)M(v)$	$(\nabla') : N(u)N(v) \leq N(kuv)$
$(\Delta_3) : uM(u) \leq M(ku)$	$(\nabla_3) : uN[N(u)] \leq kN^2(u)$
$(\Delta^2) : M^2(u) \leq M(ku)$	$(\nabla^2) : N(u^2) \leq kuN(u)$
$(\Delta_\Phi) : \Phi[M(u)] \leq M(ku)$	$(\nabla_\Phi) : \frac{N^{-1}(u)}{u} \leq \frac{kN^{-1}[\Phi(u)]}{\Phi(u)}$
$(\Delta_\Phi^*) : \Phi\left[\frac{M(u)}{u}\right] \leq \frac{M(ku)}{u}$	$(\nabla_\Phi^*) : N[\Phi(u)] \leq k \frac{\Phi(u)N(u)}{u}$

In the last two lines, $\Phi(u)$ denotes some N -function.

The assertions of the theorem for the cases of the Δ_2 - and Δ^2 -conditions were proved in ⁽¹⁾. We shall consider the remaining cases. First of all, note that in every class of equivalent N -functions there is a function with a strictly increasing continuous derivative (for example, the N -function (4)). Therefore, without loss of generality, we may assume that the derivatives $p(u)$ and $q(u)$ considered below are continuous and strictly increasing.

Lemma 1. In order that an N -function $M(u)$ satisfy the Δ' -condition, it is necessary and sufficient that its derivative $p(u)$ satisfy the analogous condition: for some k' and $u_0 > 0$,

$$p(uv) \leq k'p(u)p(v) \quad (u, v \geq u_0). \quad (5)$$

Proof. Let

$$M(uv) \leq kM(u)M(v) \quad (u, v \geq u_1). \quad (')$$

Then, by virtue of (1) and the fact that $M(u)$ satisfies the Δ_2 -condition,

$$p(uv) \leq \frac{M(2uv)}{uv} \leq k_1 \frac{M(uv)}{uv} \quad (u, v \geq u_2).$$

By virtue of (Δ') and (1),

$$p(uv) \leq kk_1 \frac{M(u)M(v)}{uv} \leq kk_1 p(u)p(v) \quad (u, v \geq \max\{u_1, u_2\}).$$

The necessity of the condition of the lemma is proved. Sufficiency is proved analogously. The lemma is proved.

It follows from (2) that $p(u)$ satisfies (5) if and only if the inverse function $q(v)$ satisfies the condition

$$q(u)q(v) \leq q(k'uv) \quad (u, v \geq v_0 = p(u_0)). \quad (6)$$

In the same way as Lemma 1, one proves:

Lemma 2. In order that $N(v)$ satisfy the ∇' -condition, it is necessary and sufficient that its derivative $p(v)$ satisfy condition (6).

Combining Lemmas 1 and 2, we obtain the assertion of the theorem for the case of the Δ' -condition. This part of the theorem was proved in [1] under the additional assumption that the N -function $N(v)$ satisfies the Δ_3 -condition.

Let us prove the assertion of the theorem for the case of the Δ_Φ -condition. Suppose that, for $u \geq u_0$,

$$\Phi[M(u)] \leq M(ku). \quad (\Delta_\Phi)$$

Let $\varphi(u)$ be the derivative of $\Phi(u)$. Then, by (1),

$$p(u)\varphi[up(u)] \leq \frac{M(2u)}{u}\varphi[M(2u)] \leq \frac{\Phi[M(4u)]}{u} \leq \frac{M(4ku)}{u} \leq 4kp(4ku).$$

Putting $k' = 4k$ and $v = p(u)$, we obtain, by (2),

$$v\varphi[vq(v)] \leq k'p[k'q(v)] \quad (v \geq v_0 = p(u_0)). \quad (7)$$

The function $M(u)$, satisfying the Δ_Φ -condition, evidently also satisfies the ∇_2 -condition. Therefore its derivative, as is easy to show, satisfies, for large u , the inequality

$$2p(u) \leq p(k''u). \quad (8)$$

It follows from (7) and (8) that, for some k_1 and v_1 ,

$$v\varphi[vq(v)] \leq p[k_1q(v)] \quad (v \geq v_1),$$

i.e.,

$$q\{v\varphi[vq(v)]\} \leq k_1q(v).$$

By (1),

$$q\left\{\frac{\Phi[N(v)]}{q(v)}\right\} \leq k_1q(v).$$

Consequently,

$$N\left\{\frac{\Phi[N(v)]}{q(v)}\right\} \leq k_1\Phi[N(v)],$$

so that

$$N\left\{\frac{v\Phi[N(v)]}{N(v)}\right\} \leq k_1\Phi[N(v)] \quad (v \geq v_1).$$

Putting $u = N(v)$, we obtain

$$N\left[\frac{N^{-1}(u)\Phi(u)}{u}\right] \leq k_1\Phi(u) \quad (u \geq u_1 = N(v_1)),$$

whence (∇_Φ) follows.

Similarly, one proves that (Δ_Φ) follows from (∇_Φ) , and the other assertions of the theorem are proved.

In [1] the question was raised whether in every class of equivalent N -functions satisfying the Δ' -condition there exists a function which satisfies this condition for all u and v . This question has a negative answer. Let

$$M(u) = (1 + u) \ln(1 + u) - u \quad (u \geq 0).$$

It is known that $M(u)$ satisfies the Δ' -condition. Suppose that there exists an N -function $\Phi(u)$ equivalent to it such that

$$\Phi(uv) \leq k\Phi(u)\Phi(v) \quad (u, v \geq 0). \quad (9)$$

By the equivalence of $\Phi(u)$ and $M(u)$, there exist k_1, k_2 and $u_0 > 0$ such that

$$M(k_1u) \leq \Phi(u) \leq M(k_2u) \quad (u \geq u_0).$$

For any fixed $u > 0$ and $v \geq \max\left\{u_0, \frac{u_0}{u}\right\}$,

$$\Phi(uv) \geq M(k_1uv) = (1 + k_1uv) \ln(1 + k_1uv) - k_1uv,$$

$$\Phi(v) \leq M(k_2v) = (1 + k_2v) \ln(1 + k_2v) - k_2v.$$

Therefore

$$\frac{\Phi(uv)}{\Phi(v)} \geq \frac{(1 + k_1uv) \ln(1 + k_1uv) - k_1uv}{(1 + k_2v) \ln(1 + k_2v) - k_2v},$$

whence it follows that

$$\lim_{v \rightarrow \infty} \frac{\Phi(uv)}{\Phi(v)} \geq \frac{k_1}{k_2} u \quad (u > 0).$$

On the other hand,

$$\frac{\Phi(uv)}{\Phi(v)} \leq \frac{k\Phi(u)\Phi(v)}{\Phi(v)} = k\Phi(u).$$

Thus,

$$\Phi(u) \geq \frac{k_1}{kk_2} u$$

for all $u > 0$, which contradicts the properties of the N -function $\Phi(u)$ (from (1) it follows that $\lim_{u \rightarrow 0} \frac{\Phi(u)}{u} = 0$).

In (1) it is proved that the N -functions $N(v)$ complementary to the N -functions $M(u)$ satisfying the Δ_3 - (respectively, Δ^2 -) condition satisfy the Δ_2 - (respectively, Δ' -) condition. These assertions follow in an obvious way from the theorem proved above.

Let us give an example of a function $M(u)$ satisfying the Δ_3 -condition, but not satisfying the ∇' -condition. Let

$$p(t) = \begin{cases} t, & \text{for } 0 \leq t \leq 2, \\ (nt)^{2(n+1)^2}, & \text{for } 2^{2^n} \leq t < 2^{2^{n+1}}. \end{cases}$$

The N -function

$$M(u) = \int_0^{|u|} p(t) dt$$

has the stated properties.

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REFERENCES

1. M. A. Krasnosel'skii, Ya. B. Rutitskii, *Convex Functions and Orlicz Spaces*, 1958.

Note: Figure translations are in progress. See original paper for figures.

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