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ON THE REALIZATION OF MONOTONE FUNCTIONS BY CIRCUITS OF FUNCTIONAL ELEMENTS

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Abstract

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CYBERNETICS AND CONTROL THEORY

V. I. REZNIK

ON THE REALIZATION OF MONOTONE FUNCTIONS BY CIRCUITS OF FUNCTIONAL ELEMENTS

(Presented by Academician M. V. Keldysh on 8 VIII 1960)

In the synthesis of circuits from given elements, the main problem is the construction of the simplest circuits realizing the functions of a certain class. We shall consider the class of monotone functions (^{4}). The degree of complexity of the circuits constructed in this class can be estimated by means of the function $L_M(n)$—the least number such that any monotone function of n arguments can be realized by a circuit containing no more than $L_M(n)$ elements. As elements we shall take functional elements realizing conjunction and disjunction. Definitions of functional elements and of circuits of functional elements are given in (^{1}).

Using E. N. Gilbert' s results on the number of monotone functions of n arguments (^{5}) and O. B. Lupanov' s theorem on obtaining lower bounds for the complexity of circuits (^{2}), it is not difficult to show that $L_M(n) \geq c_0 2^n / n^{3/2}$.

An upper bound for $L_M(n)$ can be obtained by a method analogous to Shannon' s method. Shannon' s method is based on the expansion of a function

$$f(x_1, \dots, x_n) = \bigvee_{\text{over all sets } \sigma_{k+1}, \dots, \sigma_n} x_{k+1}^{\sigma_{k+1}} \cdots x_n^{\sigma_n} f(x_1, \dots, x_k, \sigma_{k+1}, \dots, \sigma_n).$$

Circuits constructed by Shannon' s method consist of two parts: a circuit A, realizing all conjunctions $x_{k+1}^{\sigma_{k+1}} \cdots x_n^{\sigma_n}$, and a circuit B, realizing all functions of the arguments $x_1, \dots, x_k$; moreover, these parts are the same for all circuits realizing functions of n arguments, but for different functions they are connected with one another in different ways. When Shannon' s method is applied to the realization of monotone functions, instead of circuit B one may use a circuit B1, realizing only all monotone functions of the variables $x_1, \dots, x_k$. The complexity of the circuit is connected with the number of monotone functions of k variables.

The application of Shannon' s method to monotone functions makes it possible to obtain the estimate

$$L_M(n) \leq c_1 2^n \lg\lg n / n\sqrt{\lg n}.$$

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Here, however, no use is made of the fact that from a monotone function $f(x_1, \dots, x_n)$, by substitutions of constants, not all monotone functions of the arguments $x_1, \dots, x_k$ can be obtained. The method proposed in this paper makes it possible, to a certain extent, to take this circumstance into account and to lower the upper bound to

$$L_M(n) \leq c_2 2^n \lg n / n^{2*}.$$

The circuits here consist of three parts: a circuit A^1, realizing the positive conjunctions of the variables $x_{k+1}, \dots, x_n$; a circuit B^1, realizing monotone functions of the variables $x_1, \dots, x_l$; and a circuit C, realizing (using the results delivered by the circuit B^1) monotone functions of the variables $x_1, \dots, x_k$ that are obtained from the original function as a result of substitutions of constants. The circuits A^1 and B^1 are common for all monotone functions of the arguments $x_1, \dots, x_n$, while the circuits C are different for different functions. Owing to this, it is possible to carry the decomposition of functions further than in the case

Shannon' s method (to functions of l variables). In an analogous way the same upper estimate for $L_M(n)$ can be obtained for other circuits with oriented elements (for example, for relay-contact circuits).

Extending the method to the case of π-circuits (formulas) makes it possible to obtain, for the corresponding complexity function, the estimate

$$L^\pi_M(n) \lesssim c_3 \frac{2^n}{n^{1/4}} \lg n \quad \left(\text{the lower estimate } L^\pi_M(n) \gtrsim c_4 \frac{2^n}{n^{1/2} \lg n}\right).$$

Let us call an expression of the form $(x_1 \vee \sigma_1) \cdots (x_n \vee \sigma_n)$, for an arbitrary fixed set $\sigma_1, \dots, \sigma_n$, an elementary conjunction. Any monotone function $f(x_1, \dots, x_n)$ of n arguments can be represented in disjunctive normal form not containing negations (^{4}). Let the length $N(f)$ of a monotone function $f(x_1, \dots, x_n)$ be the number of terms in the corresponding DNF. $N(f) \leq 2^n/\sqrt{n}$ $(n > 2)$ (see (^{5})). Denote $2^n/N(f) = \psi(f)$ (hereafter, instead of $\psi(f)$, we shall write ψ). In the paper two cases will be considered separately:

1) $N(f) \geq 2^{n-\sqrt{n}}$ (this corresponds to $\lg \psi \leq \sqrt{n}$); 2) $N(f) < 2^{n-\sqrt{n}}$ (this corresponds to $\lg \psi > \sqrt{n}$).

Suppose we have a monotone function $f(x_1, \dots, x_n)$ given in DNF. Expand this function with respect to the last $n - k$ arguments:

$$f(x_1, \dots, x_n) =$$

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[*] This estimate, obtained in the basis $\vee, \&$, does not change, up to a constant factor, when passing to any other basis.

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$$= \bigvee_{\text{over all sets } \sigma_{k+1},\dots,\sigma_n} (x_{k+1} \vee \sigma_{k+1}) \cdots (x_n \vee \sigma_n) \cdot f_{\sigma_{k+1},\dots,\sigma_n}(x_1, \dots, x_k). \qquad (1)$$

The expansion is carried out as follows. We fix a set $\sigma_{k+1}, \dots, \sigma_n$, write the elementary conjunction $(x_{k+1} \vee \sigma_{k+1}) \cdots (x_n \vee \sigma_n)$ corresponding to this set; then we turn to the DNF corresponding to the function $f(x_1, \dots, x_n)$, and select in it all terms that contain, as a part, the conjunction of variables $(x_{k+1} \vee \sigma_{k+1}) \cdots (x_n \vee \sigma_n)$ and contain no other variables from the set $\{x_{k+1}, \dots, x_n\}$; we factor this conjunction out of the selected terms, and the expression that remains in parentheses forms a function of k arguments and is denoted by $f_{\sigma_{k+1},\dots,\sigma_n}(x_1, \dots, x_k)$.

If in the initial DNF there is no term containing the elementary conjunction $(x_{k+1} \vee \sigma'_{k+1}) \cdots (x_n \vee \sigma'_n)$, where $\sigma'_{k+1}, \dots, \sigma'_n$ is some fixed set of zeros and ones, then we shall assume

$$f_{\sigma'_{k+1},\dots,\sigma'_n}(x_1, \dots, x_k) \equiv 0.$$

If there is exactly one such term and in it variables from the set $\{x_1, \dots, x_k\}$ are absent, then we shall set

$$f_{\sigma'_{k+1},\dots,\sigma'_n}(x_1, \dots, x_k) \equiv 1.$$

All distinct functions of k arguments obtained in (1), not identically equal to zero, may be regarded as written in DNF. To a monotone function $f(x_1, \dots, x_n)$ of n arguments assign the number $B(k, f)$—the number of distinct functions of k arguments $x_1, \dots, x_k$ that are formed by the expansion (1), and let $r_i(f)$ denote the number of distinct functions of k arguments $x_1, \dots, x_k$ obtained in (1) and having i terms in their DNF ($0 \leq i \leq s$, where $s = \max N(f_{\sigma_{k+1},\dots,\sigma_n}(x_1, \dots, x_k))$ over all functions of k arguments obtained in (1)). Obviously, the equalities hold:

$$\sum_{i=0}^{s} r_i(f) = B(k, f), \qquad \sum_{i=0}^{s} i r_i(f) = N(f). \qquad (2)$$

Let us have a sequence $\alpha = \{a_1, a_2, \dots\}$ of nonnegative numbers. Denote

$$S_\alpha(m) = \sum_{i=0}^{m} i a_i;$$

$p_\alpha(N)$ is the minimal m for which

$$S_\alpha(m) \geq N(f)$$

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(for convenience of notation we shall henceforth put $N(f) = N$). Obviously, in any sum $S_\alpha(p_\alpha(N))$ one can find a number $a'_{p_\alpha(N)} \neq 0$,

that $\alpha'_{p_\alpha(N)} \leqslant \alpha_{p_\alpha(N)}$ and

$$S_\alpha(p_\alpha(N)) - (\alpha_{p_\alpha(N)} - \alpha'_{p_\alpha(N)})\, p_\alpha(N) = N. \tag{3}$$

Lemma 1. Let there exist two sequences of nonnegative numbers $\alpha = \{\alpha_1, \alpha_2, ...\}$, $\beta = \{\beta_1, \beta_2, ...\}$ such that (3) is satisfied for them and $\alpha_i \geqslant \beta_i$; then: 1) $p_\alpha(N) \leqslant p_\beta(N)$; 2)

$$\sum_{i=0}^{p_\alpha(N)-1} \alpha_i + \alpha'_{p_\alpha(N)} \geqslant \sum_{i=0}^{p_\beta(N)-1} \beta_i + \beta'_{p_\beta(N)}.$$

Lemma 2. $B(k, f) \leqslant \sum\limits_{i=0}^{p_\alpha(N)-1} C_{2^k}^i + T$, where T' and $p_\alpha(N)$ are such that

$$N = \sum_{i=0}^{p_\alpha(N)-1} i C_{2^k}^i + p_\alpha(N) T'.$$

Denote

$$\sum_{i=0}^{p_\alpha(N)-1} C_{2^k}^i + T' = R(k, N).$$

Lemma 3. $R(k+1, N) \leqslant R(k, N)$ for $N < 2^{k-1} \cdot 2^{2k}$.

Lemma 4. If $k = [\lg\psi + \lg n - \lg\lg\psi] - 4$, $\psi \to \infty$ $(n \to \infty)$, then for sufficiently large n the relation

$$C_{2k}^{[n/\lg\psi]} < 2^{(1-2/\lg\psi)n}$$

holds.

Lemma 5. If $\lg\psi \leqslant \sqrt{n}$, $k = [\lg n + \lg\psi - \lg\lg\psi] - 4$, $\psi \to \infty$ $(n \to \infty)$, then

$$R(k, N) \leqslant \frac{2^n}{n\psi} \lg\psi.$$

For the proof it suffices in the expression

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Fig. 1

Figure 1: Fig. 1

$$R(k,N) = \sum_{i=0}^{t-1} C_{2^k}^i + \sum_{i=t}^{p_\alpha(N)-1} C_{2^k}^i + T' < (t-1)C_{2^k}^{t-1} + \sum_{i=t}^{p_\alpha(N)-1} C_{2^k}^i + T'$$

to put $t = \left[\frac{n}{\lg\psi}\right] + 1$ and, using (3), to show that

$$T' + \sum_{i=t}^{p_\alpha(N)-1} C_{2^k}^i \leqslant \frac{2^n}{\psi(t)} < \frac{2^n \lg\psi}{n\psi};$$

further, from Lemma 4 it follows that

$$\frac{(t-1)C_{2^k}^{t-1}}{2^n \lg\psi/n\psi} \to 0 \quad \text{as } n \to \infty.$$

Therefore

$$R(k,N) \leqslant \frac{2^n}{\psi n} \lg\psi.$$

Fig. 1

Denote by $L_M(n,\psi)$ the least number such that a monotone function $f(x_1, \dots, x_n)$ of length $2^n/\psi$ ($\lg\psi \leqslant \sqrt{n}$, $\psi \to \infty$ $(n \to \infty)$) can be realized by a circuit containing no more than $L_M(n,\psi)$ functional elements.

Lemma 6. $L_M(n,\psi) \leqslant a2^n \lg\psi/n\psi$, where a is a certain constant.

Proof. Consider an arbitrary monotone function of length $2^n/\psi$. It can be represented in the form (1). We shall construct a circuit realizing the function $f(x_1, \dots, x_n)$ from separate blocks (Fig. 1), each of which will be constructed from functional elements realizing conjunction and disjunction. We give a description of these blocks.

1. Block B' has $l = [\lg n]$ inputs $x_1, \dots, x_l$ and is a universal multi-output network realizing all monotone functions of l arguments; moreover, its complexity does not exceed $C_5 \cdot l^{C_1[l/2]}$.* (3).
2. Block C realizes the distinct functions of k arguments obtained in (2) (different from constants) by a method analogous to the method of G. N. Povarov

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* By the complexity of a circuit we mean the number of functional elements.

(6). Let these functions be

$$\{f_1, f_2, \dots, f_{B(k,f)}\}. \tag{4}$$

Since these functions are obtained by the decomposition (1) of the function $f(x_1, \dots, x_n)$, it follows, by Lemma 2, that $B(k, f) \le R(k, N)$. Apply to each function $f_i(x_1, \dots, x_k)$ from the system (4) the decomposition (1) with respect to the last argument x_k:

$$f_i(x_1, \dots, x_k) = x_k \cdot f_i'(x_1, \dots, x_{k-1}) \vee f_i(x_1, \dots, x_{k-1}, 0)$$

$$(1 \le i \le B(k, f)).$$

As a result, we select a system of distinct functions of $k - 1$ arguments:

$$\{f_1^*, f_2^*, \dots, f_{B(k-1,f)}^*\}. \tag{5}$$

Obviously, the system (6) can also be obtained by decomposing the function $f(x_1, \dots, x_n)$ with respect to $n - k + 1$ arguments. Therefore

$$B(k-1, f) \le R(k-1, N).$$

Continuing these decompositions until a system of distinct functions of $l + 1$ arguments is obtained, the following inequalities hold:

$$B(i, f) \le R(i, N), \quad l + 1 \le i \le k.$$

Since $l = [\lg n]$ and Lemma 3 is applicable, we have

$$R(k, N) \ge R(k-1, N) \ge \dots \ge R(l+1, N).$$

The number of obtained systems of distinct functions is $k - l$; therefore the complexity of block C does not exceed

$$R(k, N) \cdot (k - l) \cdot a,$$

where a is a constant.

3. The block A' has inputs $x_{k+1}, \dots, x_n$ and realizes conjunctions of the form

$$(x_{k+1} \vee \sigma_{k+1}) \cdots (x_n \vee \sigma_n)$$

$$((\sigma_{k+1}, \dots, \sigma_n) \ne (1, \dots, 1)).$$

The complexity of block A' does not exceed 2^{n-k}.

4. The block D realizes the function $f(x_1, \dots, x_n)$ from the functions realized by the blocks A' and C. The complexity of block D does not exceed $2 \cdot 2^{n-k}$. Hence it follows that the complexity of a circuit realizing an arbitrary monotone function $f(x_1, \dots, x_n)$ of length

$$N = \frac{2^n}{\psi} \quad (\psi \leq 2^{\sqrt{n}})$$

does not exceed

$$a \cdot R(k, N) \cdot (k-l) + 3 \cdot 2^{n-k} + C_5 \cdot l^{C^{[l/2]}}.$$

Put

$$k = [\lg n + \lg \psi - \lg \lg \psi] - 4, \qquad l = [\lg n].$$

Then $k - l < \lg \psi$, and, by Lemma 5,

$$R(k, N) \preccurlyeq \frac{2^n}{n\psi} \lg \psi$$

and

$$L_M(n, \psi) \preccurlyeq a \frac{2^n}{n\psi} (\lg \psi)^2 + 3 \cdot 2^5 \cdot \frac{2^n}{n\psi} \lg \psi + C_5 \cdot l^{C^{[l/2]}}.$$

It is easy to verify that

$$\frac{C_5 \cdot l^{C^{[l/2]}}}{2^n \lg^2 \psi / n^{3/2}} \to 0 \quad \text{as } n \to \infty.$$

Therefore, finally, we have

$$L_M(n, \psi) \preccurlyeq a \frac{2^n}{n\psi} \lg^2 \psi.$$

Theorem.

$$L_M(n) \preccurlyeq b \frac{2^n}{n^{3/2}} \lg^2 n.$$

Proof. 1) Let

$$N(f) \geq 2^{n-\sqrt{n}}.$$

Then, by Lemma 6,

$$L_M(n, \psi) \preccurlyeq a \frac{2^n}{n\psi} \lg^2 \psi$$

and, taking into account that $\psi \geq \sqrt{n}$ (see (5)), we have

$$L_M(n, \psi) \preccurlyeq b \frac{2^n}{n^{3/2}} \lg^2 n.$$

2) Let

$$N(f) < 2^{n-\sqrt{n}}.$$

The realization of the function f is carried out by its DNF. The complexity of the corresponding circuit does not exceed

$$n \cdot 2^{n-\sqrt{n}}.$$

From 1) and 2) it follows that

$$L_M(n) \preccurlyeq b\frac{2^n}{n^{3/2}}\lg^2 n.$$

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Note: Figure translations are in progress. See original paper for figures.

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