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# MATHEMATICS

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**Abstract**

**Full Text**

**MATHEMATICS**

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**GENERALIZED CHEBYSHEV NETS AND MANIFOLDS OF BOUNDED CURVATURE**

*(Presented by Academician V. I. Smirnov on 9 January 1961)*

1. In the plane of variables  $\{u, v\}$ , consider the square  $D$ ,  $0 < u < a$ ,  $0 < v < a$ . Let  $\alpha(u)$ ,  $\beta(v)$  be functions of bounded variation, left-continuous on the interval  $[0, a)$ , and let  $\mu(M)$  be a completely additive function of Borel sets from the square  $D$ , having bounded variation. Denote by  $Q_{u_0, v_0}$  the rectangle  $0 < u < u_0$ ,  $0 < v < v_0$ .

A function  $f(u, v)$  defined in the square  $D$  will be called a function of bounded variation if it admits the representation

$$f(u, v) = \alpha(u) + \beta(v) + \mu(Q_{u,v}),$$

where the properties of the functions  $\alpha(u)$ ,  $\beta(v)$ ,  $\mu(Q_{u,v})$  are described above.

We shall say that a certain function of bounded variation  $f(u, v)$  generates in the square  $D$  a generalized Chebyshev line element

$$ds^2 = du^2 + 2 \cos f(u, v) du dv + dv^2, \tag{1}$$

if the function  $f(u, v)$  admits the representation

$$f(u, v) = -A + \alpha(u) + \beta(v) + \mu(Q_{u,v}),$$

where: 1)  $0 < A = \alpha(0) = \beta(0) < \pi$ ; 2)  $0 < \alpha(u) < \pi$ ,  $0 < \beta(v) < \pi$  for  $0 \leq u < a$ ,  $0 \leq v < a$ ; 3) for all  $(u, v) \in D$  we have  $0 < f(u, v) < \pi$ .

If one considers a two-dimensional Riemannian manifold on which a Chebyshev coordinate net has been introduced, then in this net the line element of the manifold has the form (1), where  $f(u, v)$  is at least twice continuously differentiable;  $f(u, v)$  is, obviously, the angle between the coordinate lines at the point of the manifold with coordinates  $(u, v)$ . The geometric meaning of the quantities  $A$ ,  $\alpha(u)$ ,  $\beta(v)$ ,  $\mu(Q_{u,v})$  is as follows:

$$A = f(0, 0), \quad \alpha(u) = f(u, 0), \quad \beta = f(0, v),$$

$$\mu(Q_{u,v}) = \int_0^u \int_0^v K(u,v) d\sigma,$$

where  $K(u,v)$  is the Gaussian curvature, and  $d\sigma$  is the area element of the manifold under consideration. It is obvious that in this case the functions  $\alpha(u)$ ,  $\beta(v)$ ,  $\mu(Q_{u,v})$ ,  $f(u,v)$  satisfy conditions 1)–3).

**2.** A curve  $L$  in  $D$  will be called smooth if there exists a parametrization of it  $u = u(t)$ ,  $v = v(t)$ ,  $a \leq t \leq b$ , such that  $u(t), v(t) \in C^1$  and everywhere on  $[a, b]$ ,  $u'^2 + v'^2 \neq 0$ .

**Theorem 1.** Let a generalized Chebyshev line element be defined in  $D$ . Then along every smooth curve  $L$  the function

$$\sqrt{u'^2(t) + 2 \cos f(u(t), v(t)) u'(t)v'(t) + v'^2(t)}$$

is summable on  $[a, b]$ .

We define the length of the curve  $L$  with respect to the generalized Chebyshev line element under consideration by the formula

$$s_f(L) = \int_a^b \sqrt{u'^2 + 2 \cos f(u, v) u'v' + v'^2} dt.$$

Let  $A$  and  $B$  be two points of the square  $D$ . Put

$$\rho_f(A, B) = \inf s_f(L), \tag{2}$$

where the greatest lower bound is taken over all smooth curves  $L \subset D$  with endpoints at  $A$  and  $B$ . The function of a pair of points  $\rho_f(A, B)$  satisfies, as is not difficult to see, the axioms of a metric space. Moreover, this metric is evidently intrinsic.

In what follows we shall say that the generalized Chebyshev line element (1) generates the intrinsic metric (2).\*

Thus, on the square  $D$ , by means of the metric (2), the generalized Chebyshev line element generates a certain two-dimensional metric manifold, which we shall denote by  $D_f$ .

**Theorem 2.** Let a generalized Chebyshev line element (1) be given in  $D$ . Then the manifold  $D_f$  generated by this element is a manifold of bounded curvature in the sense of A. D. Aleksandrov (see (1)); moreover, the set function  $\mu(M)$  coincides with the intrinsic integral curvature of this manifold.

3. Theorem 2 admits a converse in the following sense:

**Theorem 3.** Let  $R$  be a manifold of bounded curvature and let  $G \subset R$  be a domain homeomorphic to the square  $D$ , with  $\omega^+(G) < \pi/2$ ,  $\omega^-(G) < \pi/2$  ( $\omega^+$  and  $\omega^-$  are the positive and negative parts of the intrinsic integral curvature of the manifold  $R$ ). Then in  $G$  one can introduce a coordinate net with a generalized Chebyshev line element (1), which generates a metric manifold  $G_f$  isometric to the domain  $G$ . In this case one may assume that  $A = \pi/2$ .

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### CITED LITERATURE

<sup>1</sup> A. D. Aleksandrov, DAN, 60, No. 9 (1948).

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\* According to A. D. Aleksandrov (<sup>1</sup>), the metric  $\rho_f(A, B)$  is called intrinsic if  $\rho_f(A, B) = \inf s(L)$  over all smooth curves  $L$  in  $D$  with endpoints  $A$  and  $B$ , where the length of the curve  $s(L)$  is defined as the least upper bound of the sums

$$\sum_{i=0}^{n-1} \rho_f(X_i, X_{i+1});$$

$A = X_0, X_1, \dots, X_n = B$  is a system of successive points on the curve  $L$ .

*Note: Figure translations are in progress. See original paper for figures.*

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