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**Abstract**

**Full Text**

**MATHEMATICS**

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**ON BOUNDARY-VALUE PROBLEMS FOR QUASILINEAR PARABOLIC SYSTEMS OF EQUATIONS AND ON THE CAUCHY PROBLEM FOR HYPERBOLIC EQUATIONS**

*(Presented by Academician S. L. Sobolev on 22 V 1961)*

In the first part, a mixed boundary-value problem is studied for quasilinear parabolic systems of equations whose spatial part is a strongly elliptic quasilinear operator <sup>(1)</sup>. In the second part, by means of an analogue of the Galerkin method, the existence of a solution of the Cauchy problem for a linear hyperbolic equation of arbitrary order is proved.

1. Let there be given in the cylindrical domain  $\Omega = G \times (0 < t < T)$  a quasilinear parabolic system

$$Mu \equiv \frac{\partial u}{\partial t} + \sum_{|\alpha|, |\gamma| \leq m} (-1)^{|\alpha|} D_\alpha A_\alpha(x, t, D_\gamma u) \equiv \frac{\partial u}{\partial t} + L(u) = h, \quad (1)$$

where  $x = (x_1, \dots, x_n)$ ,  $u = (u^1, \dots, u^N)$ ,  $h = (h^1, \dots, h^N)$ ,  $D_\alpha = \partial^{|\alpha|} / \partial x_{\alpha_1} \dots \partial x_{\alpha_n}$ ,  $\alpha = (\alpha_1, \dots, \alpha_n)$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_n$ , with initial and boundary conditions:

$$u|_{t=0} = \Psi(x), \quad u|_\Gamma = \varphi(x', t), \dots, \quad D_\omega u|_\Gamma = \varphi_\omega(x', t), \quad (2)$$

$x' \in \Gamma$ ,  $|\omega| \leq m - 1$ ;  $\Gamma$  is the boundary of  $G$ ;  $0 < t < T$ . Suppose that the spatial operator  $L(u)$  in (1), for each  $t$ ,  $0 < t < T$ , is strongly elliptic in the sense of <sup>(1)</sup>, but instead of the definiteness of its variation assumed in <sup>(1)</sup>, only the semiboundedness of this variation is required, for example in the form

$$\begin{aligned} A(w; v, v) &\equiv \sum_{|\alpha|, |\beta| \leq m} (A_{\alpha\beta}(x, t, D_\gamma w) D_\beta v, D_\alpha v) \geq \\ &\geq c^2 \left( \sum_{j, |\alpha|=m} |D_\alpha w^j|^{\delta_j} |D_\alpha v^j, D_\alpha v^j| \right) - K(v, v), \end{aligned} \quad (3)$$

where  $A_{\alpha\beta} = \partial A_\alpha / \partial D_\beta u$ ;  $(, )$  is the scalar product with respect to  $x$ ;  $w(x, t)$ ,  $v(x, t)$  are arbitrary sufficiently smooth functions, with  $v \in V_0$ , i.e. satisfying on  $\Gamma$  the homogeneous boundary conditions (2) (we denote them by  $(2_0)$ ); the constants  $c^2$  and  $K$  do not depend on  $w$ ,  $v$ , or  $t$ . Condition (3) is analogous to condition  $(I_1)$  of <sup>(1)</sup>. Analogously one formulates a condition of type  $(I_2)$  of <sup>(1)</sup>, or another type of semiboundedness condition analogous to item 3 of <sup>(1)</sup>.

By a solution of problem (1), (2) we understand a function  $u(x, t)$  from the corresponding space  $U$ , satisfying conditions (2) and the relation

$$\left[ \frac{\partial u}{\partial t}, v \right] + \sum [A_\alpha(x, t, D_\gamma u), D_\alpha v] = [h, v] \quad (4)$$

for every function  $v \in V_0$  (or  $v \in V'_0$ ,  $V'_0 \subset V_0$ );  $[, ]$  is the integral over  $\Omega$ . The spaces  $U$  and  $V_0$  are chosen depending on the orders of growth of  $A_\alpha$  with respect to  $D_\gamma u$ . For example,  $U$  consists of functions  $u(x, t)$  for which the norms are finite

$$\left\| \frac{\partial u}{\partial t} \right\|_2 + \sum \|D_\alpha u\|_p < +\infty \quad (|\alpha| \leq m), \quad (5)$$

where the norm  $\| \cdot \|$  is taken over  $\Omega$ , and  $V_0$  consists of functions  $v(x, t)$  satisfying on  $\Gamma$  the conditions  $(2_0)$ , with finite norms  $\|D_\alpha v\|_p$ ,  $|\alpha| \leq m$ .

Let  $f(x, t)$  be a function defined in  $\Omega$  and satisfying the initial and boundary conditions (2). Then, putting  $u = f + z$ , where  $u$  is a solution of problem (1), (2) in the sense of (4), we find that  $z \in V_0$  and  $z|_{t=0} = 0$ ; moreover, according to (4),

$$\left[ \frac{\partial z}{\partial t}, v \right] + [A_\alpha(x, t, D_\gamma(f + z)) - A_\alpha(x, t, D_\gamma f), D_\alpha v] = [h, v] - \left[ \frac{\partial f}{\partial t}, v \right] - [A_\alpha(x, t, D_\gamma f), D_\alpha v]. \quad (6)$$

**Theorem 1.** *If  $Lu$  is a semibounded strongly elliptic operator (see (1) and, for example, (3)), then for any conditions (2) and right-hand side  $h(x, t)$  possessing certain smoothness conditions (finiteness of some norms), problem (1), (2) has, and moreover uniquely, a solution.*

For brevity we give the idea of the proof only in the special case when  $A_\alpha$  are polynomials in  $D_\gamma u$  of order  $2l + 1$  with constant coefficients, and

$$A(w; v, v) \geq \gamma^2 \sum_\beta \left( \sum_\alpha |D_\alpha w|^{2l} D_\beta z, D_\beta z \right) \quad (|\alpha| = m, |\beta| = m). \quad (7)$$

The general case is treated analogously, with some complications. In the case under consideration it is required that  $\frac{\partial h}{\partial t}, \frac{\partial}{\partial t} M(f) \in \mathcal{L}_2(\Omega)$ ,  $(h - M(f))|_{t=0} =$

$0, \psi(x) \frac{\partial h}{\partial x_i}, \psi(x) D_\alpha f, |\alpha| \leq m + 2^*$ , be summable over  $\Omega$  with a certain power, where  $\psi(x)$  is any function vanishing on  $\Gamma$  together with several derivatives. We note that in other cases the conditions on  $h$  and  $f$  are, as a rule, less restrictive. In  $\Omega$  define the differential operator

$$B(z) \equiv \Lambda z e^{-\lambda t} - \psi(x) \Delta z e^{-\lambda t} - \frac{\partial}{\partial t} \left[ (T - t) \frac{\partial}{\partial t} (z e^{-\lambda t}) \right], \quad (8)$$

where  $\Lambda$  and  $\lambda$  are sufficiently large numbers. Let  $\{z_k(x)\}$  be a system of functions in  $G$  such that, by means of linear combinations of functions of the form  $B(z_k(x) d_k(t))$ , where  $d_k(t)$  are arbitrary smooth functions of  $t$ ,  $d_k(0) = 0$ , one can approximate in the norm  $\|v\| = \sum \|D_\alpha v\|_{2l+2}, |\alpha| \leq m$ , any function  $v \in V_0 \subset W_{2l+2}^m$ . The existence of such a system  $\{z_k(x)\}$  is easily established by the method of separation of variables with respect to  $x$  and  $t$  for the operator  $B$  and Lemma 2 of (1). The approximate solution of problem (1), (2), or, equivalently, (6), is sought in the form  $\tilde{z}_n(x, t) = \sum C_{in}(t) z_i(x), i \leq n$ , and the coefficients  $C_{in}(t)$  are determined from the following moment equations:

$$\left( B^* \frac{\partial \tilde{z}_n}{\partial t}, z_k(x) \right) + (B^* (L(f + \tilde{z}_n) - L(f)), z_k(x)) = (B^* (h - M(f)), z_k(x)) \quad (9)$$

under the boundary conditions

$$C_{in}(0) = C'_{in}(0) = 0 \quad (i = 1, \dots, n); \quad C_{in}|_{t=T} \text{ is bounded.} \quad (10)$$

(9) is a system of ordinary equations of third order in  $t$ , degenerating at  $t = T$ .

\* These conditions on  $f(x, t)$ , obviously, impose certain restrictions on the smoothness of the boundary and initial data in (2) and on their compatibility on  $\Gamma$  at  $t = 0$ .

**Lemma 1.** Problem (9), (10) has at least one solution  $C_{in}(t)$ . For the corresponding approximations  $\tilde{z}_n(x, t)$  the norms are uniformly bounded (in the case (7)):

$$\|D_\alpha \tilde{z}_n\|_{2l+2} + \left\| \frac{\partial \tilde{z}_n}{\partial t} \right\|_2 \leq K, \quad \left\| \frac{\partial}{\partial x_i} (D_\alpha \tilde{z}_n)^{l+1} \right\|'_2 + \left\| \frac{\partial}{\partial t} (D_\alpha \tilde{z}_n)^{l+1} \right\|'_2 \leq K, \quad |\alpha| \leq m, \quad (11)$$

where  $\| \cdot \|$  is the norm taken over  $\Omega$ , and  $\| \cdot \|'$  is over  $\Omega' = G' \times (0 < t < T)$ ,  $\overline{G'} \subset G$ .

The first part of the lemma is proved by reducing the problem to a system of integral equations with respect to  $\partial C_{in}/\partial t$ ; moreover, the corresponding operator has the form  $E + V$ , where  $V$  is a completely continuous operator. It is

shown that on a sphere of sufficiently large radius in the corresponding Hilbert space the index of the covering of zero under the mapping  $E + V$  is equal to one. To prove the estimates (11), we multiply both parts of (9) by  $C_{in}(t)$ , sum over  $i = 1, \dots, n$ , and integrate with respect to  $t$ . Next we use estimate (5) (respectively (6)) from <sup>(1)</sup> and the estimate (7) given here. From the inequality thus obtained we derive the estimates (11). It follows from (11) that one can choose a subsequence from  $\{\tilde{z}_n\}$  (we denote it also by  $\{\tilde{z}_n\}$ ) for which, weakly,

$$\partial \tilde{z}_n / \partial t \rightarrow \partial z / \partial t, \quad (D_\alpha \tilde{z}_n)^r \rightarrow (D_\alpha z)^r, \quad r \leq 2l + 2,$$

and moreover  $D_\alpha \tilde{z}_n$  converges almost everywhere to  $D_\alpha z$  (see <sup>(1,2)</sup>),  $|\alpha| \leq m$ .

We multiply both parts of (9) by arbitrary functions  $d_k(t)$ ,  $d_k(0) = 0$ , integrate with respect to  $t$  from 0 to  $T$ , transfer  $B^*$  to the second factor, and pass to the limit as  $n \rightarrow \infty$ . We obtain relation (6), in which  $B(z_k(x)d_k(t))$  is substituted for  $v$ . Since by linear combinations of such functions one can approximate any function  $v \in V_0$ , the existence of the desired function  $z(x, t)$  is proved, and with it also of the function

$$u(x, t) = f(x, t) + z(x, t),$$

which solves problem (1), (2).

The uniqueness of the solution obtained is established analogously to <sup>(2)</sup>. We note that, by virtue of the last estimates (11), the constructed function  $z(x, t)$ , and together with it the function  $u(x, t)$ , admits derivatives of order  $(m + 1)$  inside  $\Omega$ .

## 2. Cauchy problem for hyperbolic equations. Let

$$a(u) = h \tag{12}$$

be a normal hyperbolic equation in the sense of I. G. Petrovskii (see <sup>(4-6)</sup>) of order  $m + 1$ , with coefficient of  $\partial^{m+1}u/\partial t^{m+1}$  equal to one. For brevity we prescribe homogeneous initial conditions:

$$u|_{t=0} = 0, \dots, \partial^m u / \partial t^m |_{t=0} = 0. \tag{13}$$

According to a well-known remark of I. G. Petrovskii <sup>(4)</sup>, it is enough to prove the existence of a solution of problem (12), (13) under periodic boundary conditions:

$$D_\beta u|_{x=0} = D_\beta u|_{x=2\pi}, \quad |\beta| \leq m. \tag{14}$$

Let  $b(u)$  be an operator separating  $a(u)$  in the sense of Leray <sup>(5)</sup> ( $b(u) = \partial a(u) / \partial D_{tu}$ ). The existence of a solution of problem (12), (13), (14), for example for  $0 < t < T$ , can be established with the aid of an analogue of Galerkin's

method. To this end, we seek the approximate solution  $u_r(x, t)$  of the problem in the form

$$u_r(x, t) = \sum C_{\alpha r}(t) z_{\alpha}(x),$$

where

$$\alpha = (\alpha_1, \dots, \alpha_n), \quad |\alpha| \leq r, \quad r = 1, 2, \dots, \\ z_{\alpha}(x) = \exp i(\alpha_1 x_1 + \dots + \alpha_n x_n).$$

The coefficients  $C_{\alpha r}(t)$  are determined so that

$$[\varphi(t)a(u_r), b(z_{\gamma}(t)d_{\gamma}(t))] = [\varphi(t)h(x, t), b(z_{\gamma}(x)d_{\gamma}(t))], \quad |\gamma| \leq r, \quad (15)$$

where

$$\varphi(t) = e^{-\lambda t} - e^{-\lambda T} \quad (0 < t < T);$$

$\lambda$  is a sufficiently large number;  $d_{\gamma}(t)$  are arbitrary smooth functions of  $t$  satisfying homogeneous conditions at  $t = 0 \dots$

(13). Conditions (15) lead to a system of order  $(2m + 1)$  with respect to  $C_{\alpha r}(t)$ :

$$(b^*(\varphi a(u_r)), z_{\gamma}(x)) = (b^*(\varphi h), z_{\gamma}(x)) \quad (16)$$

and to the boundary conditions

$$C_{\alpha r}^{(s)}(0) = 0 \quad (s = 0, 1, \dots, m),$$

$$\left( \frac{\partial}{\partial t} (\varphi(t)(a(u_r) - h)), z_{\gamma} \right) \Big|_{t=T} = 0, \dots,$$

$$\left( \frac{\partial^{m-1}}{\partial t^{m-1}} (\varphi(t)(a(u_r) - h)), z_{\gamma} \right) \Big|_{t=T} + \dots = 0 \quad (17)$$

and to conditions of boundedness of the derivatives for  $t = T$  (the system degenerates at  $t = T$ ). The boundary conditions (17) for  $t = T$  arise as conditions for the vanishing of the coefficients of  $\partial^s d_{\gamma}(t)/\partial t^s|_{t=T}$  when the operator  $b$  is transferred to the first factor in (15).

**Lemma 2.** For sufficiently large  $\lambda$ , the boundary-value problem (16), (17) has a solution, and moreover a unique one.

**Lemma 3.** If the coefficients of the operator  $a(u)$  are sufficiently smooth, then  $u_r$ , together with the derivatives  $D_t^l D_{\alpha} u_r$ ,  $l \leq m$ ,  $l + |\alpha| \leq m + p$ , have norms uniformly bounded with respect to  $r$ :  $\|D_t^l D_{\alpha} u_r\| \leq k$ .

From the sequence  $u_r$  we select a subsequence (denote it also by  $u_r(x, t)$ ) which converges weakly to  $u(x, t)$ , with  $D_t^l D_\alpha u_r \rightarrow D_t^l D_\alpha u$ . In (15), in the term with the factor  $\partial^{m+1} u_r / \partial t^{m+1}$ , we transfer one derivative with respect to  $t$  to the factor  $b(z_\gamma d_\gamma)$ , and, for fixed  $\gamma$ , pass to the limit as  $n \rightarrow \infty$ . From the resulting relation we infer that  $u(x, t)$  also has a derivative of the form  $\partial^{m+1} u / \partial t^{m+1}$  ( $\in L_2(\Omega)$ ) and satisfies the equation  $a(u) = h$ . (Here one uses the lemma that linear combinations of functions of the form  $b(z_\gamma d_\gamma)$  are dense in the corresponding space.) From the equation  $a(u) = h$ , which is satisfied by the constructed function  $u(x, t)$ , and from Lemma 3, we infer that  $u(x, t)$  also has derivatives of the form  $D_t^{l_1} D_\alpha u$  for all  $|\alpha| + l_1 \leq m + p$ .

**Theorem 2.** *The sequence of approximations  $u_r(x, t)$ , obtained by solving the system of ordinary equations (16) (of order  $2m + 1$ ) under the boundary conditions (17), converges strongly (together with the derivatives of order  $(m - 1)$ ) to the solution  $u(x, t)$  of problem (12), (13), (14). If the coefficients  $a(u)$  and  $h(x, t)$  are sufficiently smooth, the function  $u(x, t)$  admits, respectively, derivatives of higher orders.*

The uniqueness of  $u(x, t)$  follows from the energy inequality. By slightly modifying the construction of the approximations  $u_r(x, t)$ , one can improve their convergence to  $u(x, t)$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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