NONNEGATIVE ADDITIVE FUNCTIONALS OF MARKOV PROCESSES

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Abstract

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NONNEGATIVE ADDITIVE FUNCTIONALS OF MARKOV PROCESSES

(Presented by Academician A. N. Kolmogorov, 18 XI 1960)

The formulation of the problems considered in this note is contained in the survey report (⁀1). For terminology and notation see ([1],2).

Let a homogeneous Markov process $X = (x_t, \zeta, \mathcal{M}_t, P_x, \theta_t)$ be given. By a **nonnegative additive homogeneous functional** of X we mean a function $\varphi_t^s(\omega)$ of an elementary event ω and $0 \le s \le t < \infty$, satisfying the following conditions:

A. φ_t^s is measurable in ω with respect to the σ-algebra generated by the events $\{x_u \in \Gamma\}$, $0 \le u \le t$.

There exists a set N, of P_x-measure 0, such that for $\omega \notin N$ the conditions B, C, D, E are fulfilled:

B. $\varphi_t^s + \varphi_t^u = \varphi_t^s$.

C. $\varphi_{t+h}^{s+h} = \theta_h \varphi_t^s$ (here, if the process is killed by time h, we put $\theta_h \varphi_t^s = 0$; thus, if the killing time $\zeta < \infty$, then $\varphi_t^s = 0$ for $\zeta \le s \le t \le \infty$).

D. $0 \le \varphi_t^s \le \infty$.

E. φ_t^s is right-continuous in t.

We shall briefly say "functional," omitting "nonnegative," etc.

If φ_t^s is a functional right-continuous also in s (this is fulfilled automatically if only $\varphi_0^0 < \infty$), then

$$f(x) = M_x \varphi_\infty^0 \tag{1}$$

is an excessive function, i.e. $f(x) \ge 0$, $T_t f(x) \equiv M_x f(x_t) \le f(x)$, $T_t f(x) \to f(x)$ $(t \to 0)$ (see ([3],1)). We shall call such a representation of an excessive function $f(x)$ a representation in the form of a generalized potential.

V. A. Volkonskii (⁀4) proved that an excessive function $f(x)$ is representable in the form of a generalized potential if it is: a) bounded; b) **purely excessive**: $T_t f(x) \to 0$ $(t \to \infty)$; c) $T_t f(x) \to f(x)$ uniformly in x as $t \to 0$. In this case the corresponding functional φ_t^s is given by the formula

$$\varphi_t^s = \lim_{h\to 0} \int_s^t \frac{f(x_u) - T_h f(x_u)}{h}\, du \tag{2}$$

in the sense of convergence in probability. (If the process is killed by time u, 0 is taken instead of the integrand.)

We shall consider only the so-called **standard** Markov processes (for the definition see, for example, (˜4)). Diffusion processes and the majority of other classes of processes considered are standard.

Theorem 1. Let the process X be such that every bounded purely excessive function can be represented in the form of a generalized potential, and moreover the corresponding functional is given by formula (2). Then, in order that a finite excessive function f be representable in the form of a generalized potential, it is necessary and sufficient that

$$\mathbf{M}_x f(x_{\tau\{f>n\}}) \to 0 \qquad (n \to \infty) \tag{3}$$

for all x. Here $\tau\{f > n\}$ is the time of first hitting the set $\{x : f(x) > n\}$; the function under the expectation sign is taken to be equal to zero if the process does not hit this set.

The proof of necessity is very simple; it uses only the fact that $\theta_t \varphi_\infty^0 \leqslant \varphi_\infty^0$. In proving sufficiency, the functional φ_t^s is first constructed for the purely excessive function f as the limit of the functionals corresponding to the bounded purely excessive functions $f_n = \min(f, n)$. From formula (2) it follows that these functionals do not increase as n increases, starting with that n for which the process does not hit the set $\{f > n\}$. Hence there follows the existence of the limit, uniformly in s and t. Condition (3) is used in order to prove that $\mathbf{M}_x \varphi_\infty^0 = f(x)$.

For which Markov processes is the condition of Theorem 1 fulfilled? For one-dimensional diffusion processes on an interval with unattainable endpoints it is easily derived from Volkonskii' s results, since in this case excessive functions are convex functions; they are continuous in x and, consequently, automatically satisfy condition c). But for multidimensional diffusion processes discontinuous excessive functions already exist.

For simplicity, we shall formulate Theorems 2 and 3 for the case of the three-dimensional Wiener process (Brownian motion). In this case excessive functions are nonnegative superharmonic functions, and from Riesz' s theorem (see, for example, (5)) it is easy to derive that every purely excessive function is representable in the form

$$f(x) = \int \frac{1}{|x-y|}\, \mu(dy), \tag{4}$$

where $|x-y|$ is the distance between the points x, y; μ is a measure in three-dimensional space.

Theorem 2. In order that the excessive function $f(x) \neq \infty$, given by formula (4), be representable in the form of a generalized potential, it is necessary and sufficient that the measure μ be equal to zero for every set of capacity zero. It is also sufficient only that $\mu\{f=\infty\}=0$.

The proof of necessity is based on the fact that, for a fixed point x and any nonnegative bounded Borel function $g(y)$ equal to 0 outside some fixed ball, the ratio

$$\int g(y)\,\mu(dy) \quad \text{and} \quad \mathbf{M}_x \int_1^2 g(x_u)\,\varphi(du)$$

is bounded between two positive constants. Here $\varphi(du)$ is the differential of the measure on $(0,\infty]$ defined by the relation $\varphi(s,t] = \varphi_t^s(\omega)$. Thus, if a set has capacity 0, i.e. is unattainable with probability 1, then its μ-measure is equal to 0.

The proof of sufficiency is first carried out for the case when the measure μ is entirely concentrated in some ball and the function f is bounded. For smooth functions f the corresponding functional is equal to

$$-\frac{1}{2}\int_s^t \Delta f(x_u)\,du.$$

Let us write the known formula for change of variables in sto—

on a stochastic interval (see (6))

$$f(x_s) - f(x_t) + \int_s^t \operatorname{grad} f(x_u)\,dx_u = -\frac{1}{2}\int_s^t \Delta f(x_u)\,du. \tag{5}$$

For excessive functions that are not smooth, one may use this formula in order to give meaning to the integral on its right-hand side. In this case the non-negativity of the resulting functional has to be proved separately.

It has become known to the author that Meyer (P. A. Meyer) in France has also studied questions of representability of excessive functions in the form of generalized potentials*. In his works, with which the author has had occasion to become acquainted, by a completely different method and in a more general case, the sufficiency of the condition $\mu\{f=\infty\}=0$ in Theorem 2 is proved. In addition, he states without proof a theorem asserting that a condition analogous to condition (3) is necessary and sufficient for the representability of a purely excessive function in the form of a generalized potential (i.e., the representability of a bounded purely excessive function in the form of a generalized potential is proved, and not assumed).

An interesting question is which characteristics determine an additive functional uniquely (with probability 1). V. A. Volkonskii proved that if φ_t^s is continuous, $f(x) = \mathbf{M}_x \varphi_\infty^0 < C < \infty$, then the functional is uniquely determined by specifying $f(x)$. He also proved that any finite functional of the Wiener process is continuous with probability 1 (see (4)). On the basis of these results one can prove the following theorem.

Theorem 3. There is a one-to-one correspondence between finite functionals of the three-dimensional Wiener process such that

$$\varphi_\infty^s = \lim_{t\to\infty} \varphi_t^s$$

(we do not distinguish functionals that are equal to each other with probability 1), and measures μ in three-dimensional space such that there exists an increasing sequence of closed sets F_n, whose union is the whole space, and such that

$$\int_{F_n} \frac{1}{|x-y|}\, \mu(dy) < C_n < \infty,$$

and, with probability 1, there exists an n such that $x_t \in F_n$ $(0 \le t < \infty)$. The relation between the functional and the measure is the following: for any non-negative Borel function $g(y)$,

$$\mathbf{M}_x \int_0^\infty g(x_u)\, \varphi(du) = \int \frac{g(y)}{|x-y|}\, \mu(dy).$$

The functionals φ_t^s, finite with probability 1 for $s, t \in [0, \infty), (0, \infty], (0, \infty)$, are also in one-to-one correspondence with the corresponding classes of measures.

The fact that to one functional there corresponds at most one measure follows from the uniqueness of the representation (4) for a finite excessive function. The uniqueness of the functional is derived from its continuity and the uniqueness of a continuous functional with a given bounded mathematical expectation. If a measure μ satisfying the conditions of the theorem is given, the corresponding functional is constructed as the limit of functionals corresponding to the excessive functions

$$\int_{F_n} \frac{1}{|x-y|}\, \mu(dy)$$

(these functionals all coincide, beginning with that n for which $x_t \in F_n$ for $0 \le t < \infty$). The main point of the proof consists in constructing, for a given functional (generally speaking, with infinite mathematical expectation), sets F_n

* Unpublished.

such that

$$\mathbf{M}_x \int_0^\infty \chi_{F_n}(x_u)\,\varphi(du) < \infty.$$

Here χ_{F_n} is the characteristic function of the set F_n. For this, one considers the function $\Phi_\lambda(x) = \mathbf{M}_x e^{-\lambda\varphi_\infty}$, for which the formula

$$1 - \Phi_\lambda(x) = \mathbf{M}_x \int_0^\infty \Phi_\lambda(x_u)\,\varphi(du). \tag{6}$$

is derived.

It now suffices to put

$$F_n = \{x : \Phi_{\lambda_n}(x) > 1/2\}, \quad C_n = 2/\lambda_n, \quad \lambda_n \to 0.$$

Theorems 1, 2, and 3 can be generalized to functionals satisfying, instead of condition A, the weaker measurability condition, and instead of condition B, the condition:

$$\text{B}'. \quad \varphi_{t+h}^{s+h} = \theta_h \varphi_t^s$$

almost surely (the set where the relation is not fulfilled may depend on h, s, and t) (almost homogeneous functionals).

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Note: Figure translations are in progress. See original paper for figures.