

**ON THE
COMPLETENESS OF
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 $\{z^{\tau_n} \log^j z\}$ ON CURVES AND
IN DOMAINS OF THE
COMPLEX PLANE**

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Abstract

Full Text

MATHEMATICS

CHEN SE-CHANG

ON THE COMPLETENESS OF THE SYSTEM OF FUNCTIONS $\{z^{\tau_n} \log^j z\}$ ON CURVES AND IN DOMAINS OF THE COMPLEX PLANE

(Presented by Academician I. M. Vinogradov on 21 I 1961)

There are numerous works devoted to the study of the question of completeness of the system of polynomials, as well as of the system $\{z^{\tau_n}\}$, where $\{\tau_n\}$ is a sequence of numbers, on infinite curves with a weight, in infinite domains, and in finite domains with disconnected complement. In the present paper we consider the question of completeness of the system $\{z^{\tau_n} \log^j z\}$ on infinite curves with a weight and in domains of the complex plane. In the formulation of the problems, the methods of investigation, and the results, the work is most closely related to the papers (1-7).

In contrast to these works, we do not require that the sequence $\{\tau_n\}$ satisfy the condition $\tau_{n+1} - \tau_n \geq q > 0$. Moreover, our results are established under less stringent restrictions on the weight and on the curves and domains considered.

Let the sequence $\{\nu_n\}$ of complex numbers satisfy the following conditions:

1. $\lim_{n \rightarrow \infty} \frac{n}{|\nu_n|} = D_\nu,$

2. $\operatorname{Re} \nu_n > 0, \quad |\operatorname{Im} \nu_n| < C; \quad C \text{ is a constant.}$

Denote by $\{\tau_n\}$ the sequence of all distinct numbers from $\{\nu_n\}$, and by m_n ($n = 1, 2, \dots$) the number showing how many times τ_n occurs in $\{\nu_n\}$.

Let \mathcal{L} be an unbounded curve consisting of a finite number of branches going to infinity and having the following properties:

1. It contains no loops and is rectifiable in every finite part of the plane.
2. It divides the z -plane into a finite number of infinite domains G_i ($i = 1, 2, \dots, m$), each of which contains some angle Δ_i of opening π/α_i , $1/2 \leq \alpha_i < \infty$.
3. The domain G_1 contains some curvilinear angle P with vertex at the origin. Far from the origin the angle P coincides with the angle Δ_1 . Each of the sides l_1 and l_2 of the angle P intersects $|z| = r$, $0 < r < +\infty$, at only one

point. Tangents can be drawn at the origin to the sides l_1, l_2 of the angle. The angle P has opening $\geq \pi/\alpha_1$, and

$$\frac{1}{\alpha_1} > 2(1 - D_\nu). \quad (1)$$

Here the curve \mathcal{L} may consist of a finite number of connected curves.

Suppose that on \mathcal{L} a real continuous function $\rho(z)$ is given such that, for large $|z|$,

$$\rho(z) \geq p_0(|z|) = p_0(a) + \int_a^{|z|} \frac{\omega(t)}{t} dt, \quad (2)$$

where a is a constant, $\omega(t) \geq 0$, $\omega(t) \uparrow +\infty$.

Theorem 1. Under the stated conditions, if for some $\varepsilon_0 > 0$

$$\int^\infty \frac{p_0(r)}{r^{1+\omega}} dr = +\infty, \quad \omega = \max \left(\alpha_1, \alpha_2, \dots, \alpha_m, \frac{1}{\frac{1}{\alpha_1} - 2(1 - D_\nu)} + \varepsilon_0 \right), \quad (3)$$

then the system $\{z^{\tau_n} \log^j z\}$ ($n = 0, 1, \dots$; $\tau_0 = 0$; $j = 0, 1, \dots, m_n - 1$) is complete on \mathcal{L} in the class $C[p(z)]$ of functions $f(z)$ continuous on \mathcal{L} and such that $\lim_{\substack{z \rightarrow \infty \\ z \in \mathcal{L}}} e^{-p(z)} f(z) = 0$, and is complete in the sense

$$\inf_{\{Q\}} \sup_{z \in \mathcal{L}} e^{-p(z)} |f(z) - Q(z)| = 0, \quad (4)$$

where the lower bound is taken over all possible finite linear combinations of the functions $z^{\tau_n} \log^j z$.

If for the sequence $\{\nu_n\}$ the limit $\lim_{n \rightarrow \infty} \frac{n}{|\nu_n|}$ does not exist, then we assume that the following conditions are satisfied:

1. $D_{*\nu} = \lim_{n \rightarrow \infty} \frac{n}{|\nu_n|} > 0$.
2. $\operatorname{Re} \nu_n > 0, \quad |\operatorname{Im} \nu_n| < C$.
3. $\sup_{0 < \xi < 1} \overline{\lim}_{r \rightarrow \infty} \frac{N_\nu(r) - N_\nu(r\xi)}{r - r\xi} < +\infty$,

where $N_\nu(r)$ is the number of points from $\{\nu_n\}$ lying in the disk $|z| < r$.

Moreover, in the third condition of Theorem 1 concerning the curve \mathcal{L} , the angle P will be considered rectilinear with vertex at the origin, having opening π/α_1 , $1/\alpha_1 > 2(1 - D_{*\nu})$. The remaining conditions of Theorem 1 are retained.

Theorem 2. Under the stated conditions, if condition (3) is satisfied with D_ν replaced by $D_{*\nu}$, the system $\{z^{\tau_n} \log^j z\}$ is complete in the sense (4) in the class $C[p(z)]$.

In what follows, for simplicity, we shall formulate the results only for the case when for $\{\nu_n\}$ the limit $\lim_{n \rightarrow \infty} \frac{n}{|\nu_n|}$ exists.

Under the additional assumption that, for some $\varepsilon^* > 0$,

$$\int_L e^{-p(z) \frac{1}{1+\varepsilon^*}} d\sigma < +\infty,$$

the following holds:

Theorem 3. Under the conditions of Theorem 1 the system $\{z^{\tau_n} \log^j z\}$ is complete on \mathcal{L} in the class $L_p[p(z)]$ ($p > 0$) of functions $f(z)$, defined on \mathcal{L} and such that

$$\int_L e^{-p(z)} |f(z)|^p d\sigma < +\infty,$$

and is complete in the sense

$$\inf_{\{Q\}} \int_L e^{-p(z)} |f(z) - Q(z)|^p d\sigma = 0,$$

where the lower bound is taken over all possible finite linear combinations of the functions $z^{\tau_n} \log^j z$ ($n = 1, 2, \dots$; $j = 0, 1, \dots, m_n - 1$).

Now denote by $\Delta(\alpha)$ the domain enclosed between two circles intersecting at the origin at an angle π/α .

Let the set B^* consist of a finite number of domains B_i^* ($i = 1, 2, \dots, p$) of crescent type (i.e., B_i^* is topologically equivalent to a domain bounded by two circles having internal tangency), and suppose that the domains B_i^* have only one common multiple boundary point.

at the origin. Let one of the regions complementary to \bar{B}^* , say G_1^* , contain the point at infinity. Suppose that G_1^* contains some curvilinear angle P with vertex at the origin, which has property 3 of Theorem 1, and the opening of the angle P is greater than or equal to π/α_1 , $1/\alpha_1 > 2(1 - D_\nu)$. Here D_ν is the density of the sequence $\{\nu_n\}$ satisfying condition 1. Let the remaining regions G_i^* complementary to \bar{B}^* ($i = 2, 3, \dots, p+1$) contain, respectively, angles of type $\Delta(\alpha_i)$ ($i = 2, 3, \dots, p+1$).

Let $\sigma^*(r)$ be the sum of the lengths of the arcs cut off by the set B^* on $|z| = r$, and let $\sigma^*(r) \leq e^{-p_0^*(r)}$, where

$$p_0^*(r) = p_0^*(r_0) + \int_r^{r_0} \frac{\omega^*(t)}{t} dt,$$

r_0 is a constant, and $\omega^*(t) \uparrow +\infty$ as $t \rightarrow +0$.

Theorem 4. *Under the indicated conditions, if*

$$\int_0^{\infty} \frac{p_0^*(r)}{r^{1-\omega}} dr = +\infty, \quad \omega = \max(\alpha_2, \alpha_3, \dots, \alpha_{p+1}),$$

then the system $\{z^{\tau_n} \log^j z\}$ is complete on B^* in the class $L_2[B^*]$ of functions $f(z)$ analytic in B^* and such that $\iint_{B^*} |f(z)|^2 dx dy < +\infty$, and is complete in the sense

$$\inf_{\{Q\}} \iint_{B^*} |f(z) - Q(z)|^2 dx dy = 0. \quad (5)$$

Let the set B , consisting of a finite number of simply connected infinite regions B_i ($i = 1, 2, \dots, p$), divide the plane into a finite number of simply connected regions G_i ($i = 1, 2, \dots, m$), each of which contains an angle Δ_i with opening π/α_i , and let the region G_1 have property 3 of Theorem 1 with respect to the curve \mathcal{L} .

Suppose that the half-axis $[0, +\infty]$ can be divided into two sets E_1 and E so that:

- 1) for $r \in E_1$, $\sigma(r) \leq e^{-p_0(r)}$;
- 2) for $r \geq r_0$, $E(r) \leq e^{-p_0(r)}$,

where $\sigma(r)$ is the sum of the lengths of the arcs cut off by the set B on $|z| = r$, $E(r)$ is the linear measure of the intersection of E with the ray $[r, +\infty)$, and $p_0(r)$ satisfies condition (2).

Theorem 5. *Under the indicated conditions and condition (3), the system $\{z^{\tau_n} \log^j z\}$ is complete in the sense (5) on B in the class $L_2[B]$.*

Let B be a simply connected region obtained from the whole plane by deleting a finite number of nonintersecting rectilinear angles Δ_i ($i = 1, 2, \dots, m$) with opening π/α_i , and let one angle have vertex at the origin and $1/\alpha_1 > 2(1 - D_\nu)$.

Theorem 6. *Under the indicated conditions, if (3) holds, the system $\{z^{\tau_n} \log^j z\}$ is complete on B in the class $L_2[\rho(z)]$ of functions analytic in B and such that $\iint_B e^{-\rho(z)} |f(z)|^2 dx dy < +\infty$, and is complete in the sense*

$$\inf_{\{Q\}} \iint_B e^{-\rho(z)} |f(z) - Q(z)|^2 dx dy = 0.$$

We note that all the conditions appearing in the preceding theorems are, generally speaking, also necessary.

The proof of these theorems is based on the simultaneous application of the theory of differential equations of infinite order, already used by A. F. Leont'ev⁽⁷⁾ to establish the completeness of the system $\{z^{t_n}\}$, where t_n are integers, and of a certain kernel, which in the special case is represented by a Dirichlet series and in this case was used in questions of completeness by S. Mandelbrojt⁽⁵⁾.

The theorems listed above constitute, in one sense or another, a generalization of a number of theorems of the authors mentioned at the beginning.

In conclusion, I express my sincere gratitude to Prof. A. F. Leont'ev for posing the questions and for guiding this work.

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