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Abstract

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QUALITATIVE INTEGRATION OF A SYSTEM OF n DIFFERENTIAL EQUATIONS IN A DOMAIN CONTAINING A SINGULAR POINT AND A LIMIT CYCLE

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In this note a classification is given of the trajectories in a certain domain of the phase space containing the origin and a limit cycle, for the following autonomous system of n differential equations with small parameter ε:

$$\begin{aligned} \dot{x}_1 &= a(\varepsilon)x_1 - b(\varepsilon)x_2 + \sum_{\sum j_i = 2,\,3} p^1_{j_1\cdots j_n}(\varepsilon) x_1^{j_1}\cdots x_n^{j_n} + o(|x|^3), \\ \dot{x}_2 &= b(\varepsilon)x_1 + a(\varepsilon)x_2 + \sum_{\sum j_i = 2,\,3} p^2_{j_1\cdots j_n}(\varepsilon) x_1^{j_1}\cdots x_n^{j_n} + o(|x|^3), \\ \dot{\bar{x}} &= C(\varepsilon)\bar{x} + \bar{X}(\varepsilon, x_1, x_2, \bar{x}), \end{aligned} \tag{1_ε}$$

where $x = (x_1, x_2, \dots, x_n)$; $\bar{x} = (x_3, \dots, x_n)$; $\bar{X}(\varepsilon, x_1, x_2, \bar{x})$ is an $(n-2)$-dimensional vector function; $|x|$ is the Euclidean length of the vector x; the dependence of the right-hand sides on ε is continuously differentiable the required number of times.

The system (1_ε) is subject to the restrictions:

1. $a(0) = 0$, $a'(0) \neq 0$, $b(0) \neq 0$ (for definiteness we put $a'(0) > 0$ and $b(0) > 0$).
2. The eigenvalues of the matrix $C(0)$ have negative real parts.

For the formulation of the results it is convenient to consider the system (1_ε) in such coordinates that, for $\varepsilon = 0$, the first two equations contain no terms of the form $x_1 x_j$ and $x_2 x_j$ $(j = 3, \dots, n)$. The choice of these coordinates under assumptions 1 and 2 is always possible. Therefore one may assume that the system (1_ε) is already given in such coordinates.

3.

$$g = \left\{ \frac{1}{b_0}\left[(p^2_{20} + p^1_{02})p^1_{11} - (p^2_{20} + p^2_{02})p^2_{11} - 2p^1_{20}p^2_{20} + 2p^1_{02}p^2_{02}\right] + 3p^1_{30} + p^1_{12} + p^2_{21} + 3p^2_{03} \right\} < 0^*.$$

Under assumptions 1, 2, and 3 the following is true:

Theorem 1. *There exists a neighborhood of the origin of the system (1_ε), independent of ε, such that for each $\varepsilon \in (0, \varepsilon_0)$ $(\varepsilon_0 > 0)$ this neighborhood* *contains a unique periodic solution. This solution is a stable limit cycle. For* $\varepsilon = 0$ *the origin is a stable equilibrium position, and for* $\varepsilon > 0$ *it is unstable.*

If the condition $g < 0$ is replaced by the condition $g > 0$, then the following is true: in a neighborhood of the origin, independent of ε, for each

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* To shorten the notation, the argument $\varepsilon = 0$ is omitted and zeros in the lower index are not written; for example: $p^1_{20} = p^1_{200\ldots0}(0)$.

$\varepsilon \in (\varepsilon_1, 0)$, where $\varepsilon_1 < 0$, the system (1_ε) has a unique periodic solution, which is an unstable limit cycle; the equilibrium position is stable for $\varepsilon < 0$, and unstable for $\varepsilon = 0$.

The classification of trajectories has been obtained for both cases $g < 0$ and $g > 0$, but the formulation of the results for each of these cases would take too much space, while it is inconvenient to cover both cases in a single formulation, since for $g > 0$ in Lemma 1 (see below) the fixed points β^1_ε and β^2_ε have the structure of a saddle, and not of a node*, as in the case $g < 0$. We shall dwell on the case $g < 0$ as the more interesting one.

It is known ([1]) that the trajectories of the system (1_ε) issuing from the equilibrium position fill, in an ε-neighborhood of the origin, a two-dimensional manifold. It turns out that this manifold can be extended to the limit cycle. More precisely, the following holds:

Theorem 2. *For sufficiently small* $\varepsilon \in (0, \varepsilon_0)$*, where* $\varepsilon_0 > 0$*, the system* (1_ε) *has a two-dimensional invariant manifold* M^2_ε*, homeomorphic to the plane, containing the origin and the limit cycle. The part of* M^2_ε *bounded by the cycle is filled by trajectories tending to the limit cycle as* $t \to +\infty$ *and to the equilibrium position as* $t \to -\infty$*.*

It is known ([2]) that the trajectories of the system (1_ε) tending to the equilibrium position as $t \to +\infty$ fill an $(n-2)$-dimensional manifold M^{n-2}_ε**.

In some neighborhood of the origin independent of ε, the invariant manifold M^{n-2}_ε is given by the equations

$$\begin{aligned} x_1 &= \varphi_1(x_3, \ldots, x_n, \varepsilon), \\ x_2 &= \varphi_2(x_3, \ldots, x_n, \varepsilon). \end{aligned} \tag{2}$$

Theorem 3. *Under the conditions of Theorem 1 there exists a domain* U*, independent of* $\varepsilon \in (0, \varepsilon_0)$*, containing the equilibrium position and the limit cycle, such that any solution of the system* (1_ε) *beginning in the domain* $U \setminus M^{n-2}_\varepsilon$ *tends to the cycle as* $t \to +\infty$*.*

The proof of these theorems is based on studying the transformation, effected by the system (1_ε), of a certain $(n-1)$-dimensional domain.

In the variables

$$y_1 = x_1 - \varphi_1(x_3, \dots, x_n, \varepsilon),$$
$$y_2 = x_2 - \varphi_2(x_3, \dots, x_n, \varepsilon),$$
$$\bar{y} = \bar{x}$$

the manifold M_ε^{n-2} has the equations $y_1 = 0,\ y_2 = 0$ (^{2}) (see (2)). Let $y = (y_1, 0, \bar{y})$ be a point of the $(n-1)$-dimensional half-space $P_\varepsilon:\ y_1 > 0,\ y_2 = 0$. We denote by $T_\varepsilon(y)$ the first point of intersection, with P_ε, of the trajectory that begins on P_ε at the point y.

The mapping T_ε in the coordinates y has the form

$$\tilde{y}_1 = (1 + q_1\varepsilon)y_1 + q_2\varepsilon y_1^2 + g^\theta y_1^3 + f_1(y_1, \bar{y}, \varepsilon),$$

$$\tilde{\bar{y}} = \Lambda(\varepsilon)\bar{y} + \bar{f}(y_1, \varepsilon) + \bar{F}(y_1, \bar{y}, \varepsilon),$$

where $q_1 > 0$; $\theta > 0$; $\Lambda(\varepsilon)$ is an $(n-2)$-dimensional matrix; $\bar{f}(y_1, \varepsilon)$ is an $(n-2)$-dimensional vector; $f_1(0, \bar{y}, \varepsilon) = 0$ and $f_1(y_1, \bar{y}, \varepsilon)$ contains, for $\varepsilon = 0$, no terms of the form $y_1 y_j$ $(j = 3, \dots, n)$; $\bar{F}(y_1, \bar{y}, \varepsilon)$ is an $(n-2)$-dimensional vector such that

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* Here the terms saddle and node of a mapping mean, respectively, an unstable and a stable fixed point of this mapping.

** By M_0^{n-2} we mean $\lim_{\varepsilon \to 0} M_\varepsilon^{n-2}$.

that $\bar{F}(y_1, \bar{U}, \varepsilon) = 0$. By condition 2, the eigenvalues of the matrix $\Lambda(0)$ are, in modulus, less than unity.

It is not hard to see that there exists a neighborhood U_1, independent of ε, of the origin of the coordinate system (1_ε) such that, in the domain $U_1 \cap P_\varepsilon$, the mapping T_ε has a unique fixed point

$$\beta_\varepsilon^1 = (\beta_{1\varepsilon}^1, 0, \beta_{3\varepsilon}^1, \dots, \beta_{n\varepsilon}^1),$$

where $\beta_{1\varepsilon}^1$ is of order $\sqrt{-a\varepsilon/g\theta}$ in ε. The point β_ε^1 corresponds to a periodic solution of the system (1_ε). This proves the first part of Theorem 1. Let us note that g depends, among the nonlinear parts, only on those of the first two equations, and this in fact means that the existence of the cycle is determined by the nonlinear parts only of the first two equations.

The mapping T_ε is defined analogously also for $y_1 < 0$.

The assertion of Theorem 1 on the stability of the limit cycle, as well as Theorems 2 and 3, are proved with the help of the following lemmas.

Lemma 1. For sufficiently small ε there exists a smooth curve $\Gamma_\varepsilon = \{y_1, 0, \bar{y}_\varepsilon(y_1)\}$, homeomorphic to a straight line, invariant under T_ε, passing through the origin and through the points β_ε^1 and β_ε^2*, such that

$$\lim_{k\to+\infty} T_\varepsilon^k(y_1, 0, \bar{y}_\varepsilon(y_1)) = \beta_\varepsilon^1 \quad \text{for } y_1 > 0,$$

$$\lim_{k\to+\infty} T_\varepsilon^k(y_1, 0, \bar{y}_\varepsilon(y_1)) = \beta_\varepsilon^2 \quad \text{for } y_1 < 0,$$

$$\lim_{k\to-\infty} T_\varepsilon^k(y_1, 0, \bar{y}_\varepsilon(y_1)) = 0 \quad \text{for } \beta_{1\varepsilon}^2 < y_1 < \beta_{1\varepsilon}^1.$$

Lemma 2. There exists a domain U_2, independent of ε, containing the fixed points β_ε^1 and β_ε^2 of the mapping T_ε and the origin, such that, for points $(y_1, 0, \bar{y}) \in U_2 \setminus M_\varepsilon^{n-2}$, the following relations hold:

$$\lim_{k\to+\infty} T_\varepsilon^k(y_1, 0, \bar{y}) = \beta_\varepsilon^1 \quad \text{for } y_1 > 0,$$

$$\lim_{k\to+\infty} T_\varepsilon^k(y_1, 0, \bar{y}) = \beta_\varepsilon^2 \quad \text{for } y_1 < 0.$$

We outline the proof of Lemma 1. It is known that there exists a smooth curve invariant with respect to T_ε, containing the origin, such that, for $|y_1| < l\varepsilon$ (l is a constant determined by the system (1_ε)), it is given as

$$\Gamma'_\varepsilon = \{y_1, 0, \bar{y}_\varepsilon(y_1)\};$$

$T_\varepsilon(y_1, 0, \bar{y}_\varepsilon(y_1)) \in \Gamma'_\varepsilon$ if $|y_1| < l\varepsilon$ and $|\tilde{y}_1| < l\varepsilon$, and moreover $\tilde{y}_1 > y_1$ for $y_1 > 0$ and $\tilde{y}_1 < y_1$ for $y_1 < 0$.

For each ε, consideration of the half-spaces $y_1 > 0$ and $y_1 < 0$ is analogous. For definiteness, let us consider the half-space $y_1 > 0$.

Choose on $\Gamma'_\varepsilon \cap P_\varepsilon$ two points

$$y^1 = (y_1^1, 0, \bar{y}_\varepsilon(y_1^1)) \quad \text{and} \quad y^2 = (y_1^2, 0, \bar{y}_\varepsilon(y_1^2))$$

such that

$$T_\varepsilon(y^1) = y^2. \tag{3}$$

Consider the arc γ_ε of the curve Γ'_ε with endpoints y_1 and y_2. The image of γ_ε under the mapping T_ε (denote it by $T_\varepsilon(\gamma_\varepsilon)$) is also a smooth arc, and (3) is satisfied. For some k, $T_\varepsilon^k(\gamma_\varepsilon)$ no longer lies on Γ'_ε. Moreover, one can show that there exists a k_0 such that

$$T_\varepsilon^{k_0}(\gamma_\varepsilon) \subset S_\varepsilon^1,$$

where S_ε^1 is an $(n-1)$-dimensional ball of radius of order ε with center at the point β_ε^1. The fixed point β_ε^1 turns out to be stable** for sufficiently small ε, and the first eigenvalue of the matrix of the linear parts of the expansion of the mapping T_ε in a neighborhood of β_ε^1 is of order $1-2q_1\varepsilon$. It corresponds to the manifold entering

* β_ε^2 is the fixed point of the mapping T_ε for $y_1 < 0$.

** For the topological structure of a neighborhood of a stable singular point of a mapping, see ([3]).

in β_ε^1. The remaining roots, for sufficiently small ε, do not change their character.

The set

$$\overline{(\Gamma'_\varepsilon \cap P_\varepsilon) \cup \bigcup_{k=1}^{k_0} T_\varepsilon^k(\gamma_\varepsilon)}$$

turns out to be homeomorphic to a segment. It is given as $\{y_1, 0, \bar{y}_\varepsilon(y_1)\}$, where $\bar{y}_\varepsilon(y_1)$ is a single-valued function and $0 \leqslant y_1 \leqslant \beta_{1\varepsilon}^1 - l_1\varepsilon$*. The closure of the set

$$(\Gamma'_\varepsilon \cap P_\varepsilon) \cup \bigcup_{k=1}^{\infty} T_\varepsilon^k(\gamma_\varepsilon),$$

obviously, is an invariant curve, and by what was said above, for sufficiently small ε it is given as $\{y_1, 0, \bar{y}_\varepsilon(y_1)\}$ for all $y_1 \in [0, \beta_{1\varepsilon}^1]$.

We do not give the proof of Lemma 2 for lack of space.

Results close to some of those published here were announced by Yu. I. Neimark in paper ([5]). Yu. I. Neimark asserts that they are proved on the basis of Theorem 1 of paper ([4]). However, the proof of this latter theorem in paper ([4]) contains an error.

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[4] Yu. I. Neimark, *Izv. Vyssh. uchebn. zaved., Radiofizika*, No. 2, 507 (1958).
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∗ l_1 is a certain positive number depending on k_0.

Note: Figure translations are in progress. See original paper for figures.