SovietRxiv

Soviet-era science, translated into English

MATHEMATICS

1961

SovietRxiv

View the original and related papers at https://sovietrxiv.org/items/ru-196101.82068

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

SovietRxiv

Abstract

Full Text

MATHEMATICS

Yu. I. GILDERMAN

ON EMBEDDING THEOREMS FOR ABSTRACT FUNCTIONS

(Presented by Academician S. L. Sobolev, 18 V 1961)

In the paper ([1]) S. L. Sobolev considered the B-space Φ_1 of abstract additive functions φ, defined on the totality of all measurable sets E from a certain domain $\Omega \subset R_n$ with values in the B-space X,

$$\|\varphi\|_{\Phi_1} = \sup_{\substack{E_1, E_2 \in \Omega \\ E_1 \cap E_2 = 0}} \|\varphi(E_1) - \varphi(E_2)\|_X.$$

From this space one singles out the subspace $\Psi_p^{(l)}$ of functions $\varphi(E)$ such that $\varphi_i(E) \in \Phi_{1+\varepsilon}$, $\varepsilon > 0$, and for all $\alpha = (a_1, a_2, \dots, a_n)$, $|\alpha| = l$, there exists a generalized derivative $\Psi^\alpha(E)$, continuous with respect to translation in the norm Φ_p, $p > 1$.

$$\|\varphi\|_{\Phi_p} = \sup_{\omega \in L_{p'}} \frac{\left\| \int_\Omega \omega(x)\, d\varphi(E) \right\|_X}{\|\omega(x)\|_{L_{p'}}}, \qquad \frac{1}{p} + \frac{1}{p'} = 1.$$

For the spaces $\Psi_p^{(l)}$ embedding theorems are proved, analogous to the corresponding theorems for the spaces of numerical functions $W_p^{(l)}$.

Theorem 1. If $lp > n$, the function $\varphi(E) \in \Psi_p^{(l)}$ is the integral of a certain continuous point function $\varphi(x)$. The modulus of continuity of $\varphi(x)$ is equal to $A|\Delta x|^\beta$, where $\beta > 1$,

$$\|\varphi(x)\|_X \le A \|\varphi(E)\|_{\Psi_p^{(l)}},$$

where A is a constant independent of φ.

Theorem 2. If $lp \le n$, the function $\varphi(E) \in \Psi_p^{(l)}$ is defined on all smooth manifolds of dimension s, where $s > n - pl$, and represents a function of a set $\varphi(I)$ belonging to Φ_q, where $s/q = n/p - l$.

SovietRxiv

Moreover, for any $q^* < q$ the function $\varphi(I)$ is continuous with respect to translation in the norm Φ_{q^*}, and

$$\|\varphi(I)\|_{\Phi_q} \leq A\|\varphi(E)\|_{\Psi_p^{(l)}}.$$

By slightly modifying the proof given in ([1]), one can obtain the indicated theorems starting from somewhat broader assumptions. Namely, for a function $\varphi(E) \in \Phi_1$, in addition to the existence of a generalized derivative $\Psi^\alpha(E) \in \Phi_p$ continuous with respect to translation, absolute continuity is required:

$$\|\varphi(E)\|_X < \varepsilon \quad \text{when } mE < \delta(\varepsilon).$$

However, embedding theorems 1 and 2 are also valid for broader spaces of abstract functions of sets.

Denote by Φ_1 the totality of additive and normal abstract set functions, i.e. such functions whose norm in the space X tends to 0 on a vanishing sequence of measurable sets.

Theorem 3. *In order that an additive abstract function $\varphi(E)$ be normal, it is necessary and sufficient that it be completely additive.*

Theorem 4. *The B-space $\overline{\Phi}_1$ with norm Φ_1 is a regular part of the space Φ_1.*

Obviously, $\overline{\Phi}_1$ is broader than the subspace of functions $\varphi(E) \in \Phi_1$ that are absolutely continuous.

Let $\varphi(E) \in \Phi_1$ and let $\omega(x)$ be a numerical stepwise bounded function taking only a finite number of values: $\omega(x) = a_k$, $x \in E_k$, $\sum_{k=1}^N E_k = \Omega$, the E_k do not intersect.

Then, by definition, set

$$\int_\Omega \omega(x)\, d\varphi(E) = \sum_{k=1}^N a_k \varphi(E_k).$$

Using the completeness of the space X, this operator can be extended to arbitrary bounded $\omega(x)$.

In this case

$$\left\| \int_\Omega \omega(x)\, d\varphi(E) \right\|_X \leq \max_{x\in\Omega} |\omega(x)|\, \|\varphi(E)\|_{\Phi_1}. \tag{1}$$

Lemma. *Let $\varphi(E) \in \overline{\Phi}_1$, $E_{k+1} \subset E_k$, $E_0 = \lim_{k\to\infty} E_k$; let $\omega(x)$ be a bounded numerical function and $\omega(x) = a_0$ for $x \in E_0$.*

SovietRxiv

Then

$$\lim_{k\to\infty}\int_{E_k}\omega(x)\,d\varphi(E)=a_0\varphi(E_0). \tag{2}$$

(In particular, E_0 may be of zero measure.)

Define $\overline{\Psi}_p^{(l)}$ as the totality of normal set functions $\varphi(E)$ having shift-continuous derivatives $\Psi^\alpha\in\Phi_p$.

From (1) and (2) it follows:

Theorem 5. *If $\varphi(E)\in\overline{\Psi}_p^{(l)}$, then it is absolutely continuous.*

Hence it follows:

Theorem 6. *If $\varphi(E)\in\overline{\Psi}_p^{(l)}$, then it is shift-continuous in the norm Φ_1 and, consequently, is the limit of integrals of abstract functions continuous at a point.*

This circumstance makes it possible to pass to the limit in the integral identity

$$\varphi_h(x)=\int_\Omega K(x,y)\,d\varphi_h(E)+\int_\Omega\sum_\alpha\frac{K_\alpha(x,y)}{r^{\,n-l}}\,d\Psi_h^\alpha(E),$$

where $\varphi_h(x)$ is the mean function for $\varphi(E)$. From the equality obtained in the limit as $h\to 0$, embedding theorems 1 and 2 follow.

If X is finite-dimensional, then the embedding operator is completely continuous. (The proof does not differ from the corresponding proof for numerical functions.)

In the case of an infinite-dimensional space X, the embedding operator, generally speaking, is not completely continuous (V. I. Kondrashov' s theorem does not hold).

Indeed, the set $\{\varphi_k(x)=i_k\}$ for $x\in\Omega$, where $i_k=(0,\dots,0,1,0,\dots)$, $k=1,2,\dots$, while bounded in the norm $\overline{\Psi}_p^{(l)}$, is at the same time not compact in $X=l_2$.

Let us note that the normality condition on $\varphi(E)$ in the definition of $\Psi_p^{(l)}$ is essential. Indeed, let $X=R_1$, $\Omega=(0,1)$, $\delta>0$, and

$$\varphi(E)=\begin{cases} mE, & \text{if } (0,\delta)\notin E \text{ for any value of } \delta,\\ mE-1, & \text{if } (0,\delta)\in E \text{ for at least one } \delta.\end{cases}$$

It is easy to see that the function $\varphi(E)$ thus constructed is additive, is not normal, belongs to $\Phi_1(\Omega)$, and has a generalized derivative of any order $l\geq 1$ equal to zero. However, for it the embedding theorems 1 and 2 do not hold, since by either of these theorems the function $\varphi(E)$ must be not only normal but also absolutely continuous.

SovietRxiv

In conclusion I express my deep gratitude to Academician S. L. Sobolev, whose attention I unfailingly enjoyed in carrying out this work.

Institute of Mathematics with Computing Center
Siberian Branch of the Academy of Sciences of the USSR

Received
13 V 1961

REFERENCES

1. S. L. Sobolev, *Fund. math.*, **47**, 277 (1959).

2. S. L. Sobolev, *Some Applications of Functional Analysis in Mathematical Physics*, L., 1950.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.