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Abstract

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ON THE ARRANGEMENT OF $(n{-}1)$-DIMENSIONAL SPHERES IN AN n-DIMENSIONAL SPHERE

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It is proved that two similarly situated systems of $(n-1)$-dimensional spheres in the n-dimensional sphere $\mathfrak{S}^n$ can be isotopically carried one into the other if each sphere is normally embedded and contains an $(n-1)$-dimensional simplex of the sphere $\mathfrak{S}^n$. Some consequences (see (4)) from Brown’ s work (2) will be applied. The restriction imposed here on the spheres is analogous to Mazur’ s condition (1).

Definitions and remarks. $\mathfrak{S}^n$ is a Euclidean sphere; all simplices considered in it are spherical. $\mathfrak{S}^{n-1}$ is the equator of the sphere $\mathfrak{S}^n$. By an **isotopy** of the sphere $\mathfrak{S}^n$ is meant a system g_t of topological transformations of $\mathfrak{S}^n$, depending continuously on the parameter t, $0 \leq t \leq 1$, with g_0 the identity.

A topological sphere S^{n-1} is **normally embedded** in $\mathfrak{S}^n$ if there exists a topological mapping

$$h : \mathfrak{S}^{n-1} \times I \to \mathfrak{S}^n,$$

where I is the interval $[-1, 1]$, such that

$$h(\mathfrak{S}^{n-1}) = S^{n-1}.$$

According to Brown (2,4,5), for a normally embedded sphere S^{n-1} there exists a topological mapping of $\mathfrak{S}^n$ onto itself under which $\mathfrak{S}^{n-1}$ is mapped onto S^{n-1}.

Lemma. *Let h topologically map $\mathfrak{S}^n$ onto itself, and let*

$$S^{n-1} = h(\mathfrak{S}^{n-1})$$

contain an $(n-1)$-dimensional simplex D^{n-1} of the sphere $\mathfrak{S}^n$. There exists an isotopy g_t of the sphere $\mathfrak{S}^n$ carrying S^{n-1} into the boundary of some n-dimensional simplex d^n, fixed outside a prescribed neighborhood of one of the regions complementary to S^{n-1}, and inside some simplex $d_1^n \subset d^n$.

Proof. Denote one of the regions bounded by S^{n-1} by U, and $h^{-1}(U)$ by V. Take in U a simplex d^n, the base d^{n-1} of which lies strictly inside D^{n-1} and

$$S^{n-1} \cap d^n = d^{n-1}.$$

The boundary S^{n-2} of the simplex d^{n-1} is, evidently, normally embedded in D^{n-1}, and hence

$$\sigma^{n-2} = h^{-1}(S^{n-2})$$

is normally embedded in $\mathfrak{S}^{n-1}$.

According to Brown, there exists a topological transformation

$$f : \mathfrak{S}^{n-1} \to \mathfrak{S}^{n-1},$$

under which $f(\sigma^{n-2})$ is the boundary of some simplex δ^{n-1}. Extend f along meridians to a transformation $\bar{f}$ of the whole sphere $\mathfrak{S}^n$, and consider the mapping

$$h_1 = h\bar{f}^{-1}.$$

Observe that

$$h_1(\mathfrak{S}^{n-1}) = S^{n-1}$$

and

$$h_1(\delta^{n-1}) = d^{n-1}.$$

Let

$$\delta^n \subset V$$

be a simplex with base δ^{n-1}. Construct a topological mapping τ of the region V into itself, carrying δ^{n-1} into the lateral surface of the simplex δ^n, and fixed on

$$\mathfrak{S}^{n-1} \setminus \delta^{n-1},$$

and, analogously, a mapping

$t : U \to U$, taking d^{n-1} to the lateral surface of d^n and fixed on $S^{n-1} \setminus d^{n-1}$.

Consider a new mapping $\varphi : \mathfrak{S}^n \to \mathfrak{S}^n$, equal to h_1 outside V and to $th_1\tau^{-1}$ on the image of τ. It is defined outside the simplex δ^n. Since τ is fixed on $\mathfrak{S}^{n-1} \setminus \delta^{n-1}$, and t is fixed on $S^{n-1} \setminus d^{n-1}$ and $h_1(\delta^{n-1}) = d^{n-1}$, we have

$$h_1(x) = th_1\tau^{-1}(x), \quad \text{if } x \in \mathfrak{S}^{n-1} \setminus \delta^{n-1}.$$

Consequently, φ maps topologically $\mathfrak{S}^n \setminus \delta^n$ onto $\mathfrak{S}^n$, and the boundary δ^n is mapped onto the boundary d^n. Extend φ inside these simplexes along their radii.

There exists an isotopy S_t which carries the ball V into the simplex d^n, and is fixed outside $\varphi^{-1}(O_\varepsilon(U))$, where ε is arbitrary, and on $\varphi^{-1}(d_1^n)$, $d_1^n \subset d^n$. The isotopy

$$g_t = \varphi S_t \varphi^{-1}$$

carries S^{n-1} to the boundary of the simplex d^n, and is fixed outside $O_\varepsilon(U)$ and on d_1^n.

Theorem. *Let a finite number of normally embedded and pairwise nonintersecting spheres S_i^{n-1}, $1 \leq i \leq k$, be situated in the sphere $\mathfrak{S}^n$. If each of them* *contains an* $(n-1)$*-dimensional simplex of the sphere* $\mathfrak{S}^n$*, then there exists an isotopy of* $\mathfrak{S}^n$ *carrying all* S_i^{n-1} *into the boundaries of* n*-dimensional simplexes.*

Proof. For each sphere S_i^{n-1} there exists a topological mapping

$$h_i : \mathfrak{S}^n \to \mathfrak{S}^n,$$

under which the equator $\mathfrak{S}^{n-1}$ is mapped onto S_i^{n-1}. Put ψ_{t0} identically for all t, and suppose that an isotopy ψ_{tj-1} has already been constructed which carries the first $j-1$ $(1 \le j \le k)$ spheres into the boundaries of simplexes and does not change the conditions for the remaining spheres. Let us retain the former notation for the images under ψ_{tj-1}.

Denote by U_j one of the domains bounded by the sphere S_j^{n-1}. Let ε be so small that $O_\varepsilon(U_j)$ contains no spheres S_i^{n-1} lying outside U_j. Take in U_j two simplexes d and $d' \subset d$, and let g_t' be the isotopy existing by the lemma, which carries S_j^{n-1} into the boundary d and is fixed outside $O_\varepsilon(U_j)$ and on d'.

Put in correspondence with each sphere S_i^{n-1} lying in U_j the simplex d_i whose boundary it is, if $i < j$, or which it contains by hypothesis, if $i > j$. Before carrying out the isotopy g_t, isotopically carry the simplexes d_i inside d', leaving the complement of U_j fixed. For this, construct finite polygonal lines l_i, pairwise nonintersecting, such that the beginning of l_i lies in d_i, and the end in d'. Let ε be so small that $O_\varepsilon(l_i)$ are pairwise nonintersecting, homeomorphic to balls, and the ends of l_i are farther from the boundary of d' than ε. Each simplex d_i, by an isotopy fixed outside a sufficiently small neighborhood of it, is mapped onto a simplex lying in $O_\varepsilon(l_i)$, and then, fixed outside $O_\varepsilon(l_i)$, onto a simplex lying in d'. Denote the isotopy thus constructed by f_t. The isotopy

$$\psi_{tj} = g_t f_t \psi_{tj-1}$$

carries the first j spheres S_i^{n-1} into the boundaries of simplexes and preserves the conditions of the theorem for the remaining spheres. After k steps the proof of the theorem is completed.

Corollary 1. *If two similar systems of* $(n-1)$*-dimensional pairwise nonintersecting spheres are given, each of which is normally embedded and contains an* $(n-1)$*-dimensional simplex of the sphere* $\mathfrak{S}^n$*, then there exists an isotopy carrying one of these systems into the other.*

This follows from the theorem proved and from the validity of the proposition for systems of boundaries of n-dimensional simplexes.

Corollary 2. *For an arbitrary finite system of pairwise nonintersecting* $(n-1)$*-dimensional spheres, polyhedral in* $\mathfrak{S}^n$*, there exists an isotopy of* $\mathfrak{S}^n$ *carrying these spheres into the boundaries of simplexes.*

By virtue of the "n-dimensional theorem of Alexander" proved by Newman (^{3}), according to which the polyhedral sphere bounds regions whose closures are homeomorphic to balls, such a sphere is embedded in S^n normally. Consequently, for systems of polyhedral spheres the conditions of the theorem proved are satisfied.

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Note: Figure translations are in progress. See original paper for figures.

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