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Abstract

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ON A BOUNDARY-VALUE PROBLEM OF LINEAR CONJUGATION

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1. Let Γ consist of $m+1$ simple closed contours and partition the plane E into two parts: a finite domain D^+ and its complement D^-. We consider the following boundary-value problem: to find functions $\varphi^+(z)$ and $\varphi^-(z)$, analytic respectively in D^+ and D^-, continuously extendable to Γ, and connected by the relation

$$\varphi^+(t) = a(t)\varphi^-(t) + b(t)\overline{\varphi^-(t)} + c(t), \qquad \varphi^-(\infty) = 0. \tag{A}$$

Let us clarify the terms. Since the homogeneous problem admits combinations of solutions only with real coefficients, linear independence will also be understood in this sense. A linear boundary-value problem (or equation) is called normally solvable if l is finite—the number of linearly independent solutions of the homogeneous problem—and p is the number of necessary and sufficient solvability conditions for the nonhomogeneous problem.

2. Theorem 1. *Let Γ consist of smooth contours; let $a(t)$, $b(t)$, $c(t)$ satisfy a Hölder condition everywhere on Γ, and let $\varkappa = \operatorname{Ind}_\Gamma a(t)$. If $\varkappa > 0$, then the homogeneous problem has $2\varkappa$ linearly independent solutions, and the nonhomogeneous problem is unconditionally solvable. For $\varkappa = 0$ the problem has a unique solution, the zero solution for the homogeneous problem. For $\varkappa < 0$ the homogeneous problem has only the zero solution, and for solvability of the nonhomogeneous problem it is necessary and sufficient that $2|\varkappa|$ real or $|\varkappa|$ complex conditions be satisfied*

$$\int_\Gamma t^k R[c(t)]\, dt = 0, \qquad k = 0, 1, \dots, |\varkappa| - 1,$$

where R is some linear operator.

The proof is obtained from the following lemmas.

Lemma 1. *If $a(t) \neq 0$, then problem (A) is normally solvable. For $\varkappa \geqslant 0$, $l \geqslant 2\varkappa$, and for $\varkappa < 0$, $p \geqslant 2|\varkappa|$.*

Problem (A) is reduced to the equivalent singular integral equation

$$\alpha_1\mu + \beta_1 S\mu + \alpha_2\overline{\mu} + \beta_2 S\overline{\mu} = \gamma, \tag{1}$$

where

$$S\mu = \frac{1}{\pi i}\int_\Gamma \frac{\mu(\tau)}{\tau - t}\,dt,$$

and (1), in turn, is reduced to an equivalent system of singular integral equations of the usual form (see (1)).

Lemma 2. *Let l^* be the number of solutions of the conjugate homogeneous problem*

$$\psi^*(t) = a^*(t)\psi^-(t) + b^*(t)\overline{\psi^-(t)}, \qquad \psi^-(\infty) = 0, \tag{A*}$$

where

$$a^* = \frac{\overline{a}}{|a|^2 - |b|^2}, \qquad b^* = \frac{-b\,\overline{t}^2}{|a|^2 - |b|^2}.$$

Then $p \leqslant l^* u$; in particular, if $l^* = 0$, then $p = 0$ and problem (A) is solvable.

Lemma 2 follows from a work of N. P. Vekua ([2]).

Lemma 3. If $|a(t)| > |b(t)|$, then for $\chi \leq 0$ $l = 0$, and for $\chi > 0$ $l \leq 2\chi$.

Lemma 3 is proved by a qualitative comparison with the Riemann problem ($b(t) \equiv 0$) ([3,5]).

Under the condition that $a(t), b(t)$ satisfy the Hölder condition with an exponent arbitrarily close to one, Lemma 3 and Theorem 1 were proved in the work of B. V. Boyarskii ([10]), by a completely different method.

Theorem 2. Let Γ consist of Lyapunov contours; $a(t)$ be continuous; $b(t)$ be measurable and bounded; $c(t) \in L_p(\Gamma)$, $p > 1$.

If

$$\sup_{t\in\Gamma}\left|\frac{b(t)}{a(t)}\right| < \frac{2}{1 + M_p}, \tag{2}$$

where M_p is the norm of the operator S in L_p, then all the assertions of Theorem 1 are valid.

The proof of the theorem is carried out analogously to the method of I. B. Simonenko ([4]); here it is assumed that $\varphi^+(z), \varphi^-(z)$ are representable by the Cauchy integral.

Theorem 3. Let Γ consist of one smooth contour; $a(t) \equiv |b(t)| > 0$; $\chi = \operatorname{Ind}_\Gamma a(t) + \operatorname{Ind}_\Gamma b(t)$; $\mu = \operatorname{Ind}_\Gamma a(t) - \operatorname{Ind}_\Gamma b(t)$, and let $a(t), b(t), c(t)$ satisfy the Hölder condition.

Then:

1) if $\chi < 0,\ \mu \le 1$, then $l = 0,\ p = |\chi| + |\mu|$;
2) if $\chi < 0,\ \mu > 1$, then $l = \mu - 1,\ p = |\chi| - 1$;
3) if $\chi \ge 0,\ \mu > 1$, then $l = \chi + \mu,\ p = 0$;
4) if $\chi \ge 0,\ \mu \le 1$, then solvability is determined from a system of $|\mu| + 1$ equations with $\chi + 1$ unknowns.

Remark. If both numbers χ, μ are even, then the case of a multiply connected domain is also investigated:

1) $\chi < 0,\ \mu \le 0,\ l = 0,\ p = |\chi| + |\mu| + 2m$;
2) $\chi < 0,\ \mu > 2(m-1),\ l = \mu - m - 1,\ p = |\chi| + m + 1$;
3) $\chi > 2(m-1),\ \mu > 2(m-1),\ l = \chi + \mu - 2m,\ p = 0$;
4) $\chi > 2(m-1),\ \mu \le 1$, solvability is determined from a system of $|\mu| + m + 1$ equations with $\chi - m + 1$ unknowns.

Suppose the condition of normal solvability $a(t) \ne 0$ is fulfilled. As is seen from Theorems 1 and 3, in problem (A) one must distinguish the cases: $|a(t)| > |b(t)|$, $|a(t)| \equiv |b(t)|$, and $|a(t)| < |b(t)|$, which we shall call, respectively, elliptic, parabolic, and hyperbolic. Theorem 2 pertains to the elliptic case, since always $M_p \ge 1$, and condition (2) takes the form of strengthened ellipticity $|b(t)| < q|a(t)|$, where $q \le 1$. If Γ is a circle and $c \in L_2(\Gamma)$, then $M_2 = 1$, and the condition takes the form of pure ellipticity. The hyperbolic, as well as the mixed, case remains unstudied. For $a(t) \equiv 0$ problem (A)

$$\varphi^+(t) = b(t)\overline{\varphi^-(t)} + c(t), \tag{3}$$

is, generally speaking, indeterminate. Indeed, if D is a disk, then (3) reduces to the form (see ([5])) $\varphi^+ = b\psi^+ + c$, and if, in addition, $b(t) = B^+(t)$, $c(t) = C^+(t)$, where $B^+(z), C^+(z)$ are holomorphic functions, then one of the functions $\varphi^+(z), \psi^+(z)$ is completely arbitrary. For $b(t) \equiv 1,\ c(t) \equiv 0$ problem (3) was studied by A. I. Markushevich ([6]): for the disk and domains bounded by lemniscates, the problem is solvable and has the indeterminate character noted above.

Consider the singular integral equation (1), which generalizes the known characteristic singular integral equation. Since equation (1) reduces to problem (A) and is equivalent to it, all the preceding results may be transferred to it. From Theorem 1 it follows immediately:

Theorem 4. Let

$$G(t) = (\overline{\alpha_1} + \overline{\beta_1})(\alpha_1 - \beta_1) - (\alpha_2 + \beta_2)(\overline{\alpha_2} - \overline{\beta_2}),$$

$$\Delta_1(t) = |\alpha_1 + \beta_1|^2 - |\alpha_2 + \beta_2|^2, \qquad \Delta_2(t) = |\alpha_1 - \beta_1|^2 - |\alpha_2 - \beta_2|^2.$$

If $G(t) \neq 0$, equation (1) is normally solvable. If the condition $\Delta_1(t)\Delta_2(t) > 0$ is satisfied and $\varkappa = \operatorname{Ind}_\Gamma G(t)$, then for equation (1) all the assertions of Theorem 1 are valid. To (1) there corresponds the integral equation of plane potential theory

$$p(t)\mu(t) + q(t)\frac{1}{\pi}\int_\Gamma \frac{\cos\widehat{(r,n)}}{r}\,\mu(\tau)\,ds = f(t).$$

If $|p(t)| > |q(t)|$, then it has a unique solution for any $f(t)$.

3. Following I. N. Vekua ([7]), consider the generalized Cauchy–Riemann system

$$\partial_{\bar z} w + A(z)w + B(z)\overline{w} = 0, \tag{4}$$

where $z = x+iy$; $\partial_{\bar z} = \frac{1}{2}(\partial_x + i\partial_y)$; $A(z), B(z)$ are given in the whole plane E and belong to the class $L_{p,2}(E)$, $p > 2$, i.e. $A(z), B(z)$ and $|z|^{-2}A(\frac{1}{z})$, $|z|^2 B(\frac{1}{z}) \in L_p(E_1)$, $p > 2$, where E_1 is the disk $|z| \leq 1$. The function $w(z)$ is called a regular solution of equation (4) if it is continuous, has the generalized Sobolev derivative $\partial_{\bar z} w$, and satisfies equation (4) almost everywhere. Since there is a one-to-one correspondence (and other deep connections) between regular solutions $w(z)$ and analytic functions $\varphi(z)$, $w(z)$ is also called a generalized analytic function.

Consider the boundary problem (A) for equation (4). In this case problem (A) has an important geometric significance in the investigation of infinitesimal bendings of surfaces glued from pieces of ovaloids ([7]); the special case $b(t) \equiv 0$ was studied by the author ([8], [9]).

Theorem 5. For the boundary problem (A) in the class of generalized analytic functions, all the results obtained above for problem (A) in the class of analytic functions are valid (only in Theorem 2 the norm of the singular Cauchy integral M_p must be replaced by the norm of its analogue $\mathfrak{M}_p$).

The proofs are obtained either directly from the fact of the one-to-one correspondence between $w(z)$ and $\varphi(z)$, or by repeating all stages of the proofs of Theorems 1, 2, 3.

4. Let an elliptic equation of second order be given,

$$\Delta u + a(x,y)u'_x + b(x,y)u'_y = 0, \tag{5}$$

where $a, b \in L_{p,2}(E)$, $p > 2$. We pose the following boundary problem: find solutions of equation (5), regular (in the sense of item 3) in D^+ and D^-, if u, u'_x, u'_y are continuous in $D^+ + \Gamma$ and $D^- + \Gamma$, and u'_x, u'_y are coupled on Γ by the conditions

$$\begin{aligned} u_x^+ &= a_1 u_x^- + b_1 u_y^- + c_1, \\ u_y^+ &= a_2 u_x^- + b_2 u_y^- + c_2, \end{aligned} \tag{6}$$

where $a_1, b_1, a_2, b_2, c_1, c_2$ are given real functions. For definiteness of the problem it is necessary to prescribe a condition at infinity. We take it in the form:

$$|u'_x|,\ |u'_y| \le K|z|^{-\varepsilon}, \qquad K, \varepsilon > 0.$$

The homogeneous problem (6) always has the solution $u(x,y) \equiv \text{const}$, which we call trivial and do not take into account further.

Putting $u'_x - iu'_y = w(z)$, we obtain for $w(z)$ equation (4), while the boundary conditions (6) are transformed into (A). Having found $w(z)$, we return to $u(x,y)$ by the formula

$$u(x,y) = u(\infty) - \frac{1}{\pi}\iint_E \frac{\overline{w(\zeta)}}{\zeta - z}\,d\xi\,d\eta \qquad (\zeta = \xi + i\eta).$$

All the results obtained above carry over to problem (6). Denote

$$\Delta(t) = \begin{vmatrix} a_1(t) & b_1(t) \\ a_2(t) & b_2(t) \end{vmatrix}, \qquad G(t) = \frac{1}{2}\{[a_1(t) + b_2(t)] + i[b_1(t) - b_2(t)]\}.$$

If $G(t) \neq 0$, then the boundary-value problem is normally solvable. We shall call the boundary conditions (6) conditions of elliptic, parabolic, or hyperbolic type if, respectively, $\Delta(t) > 0$, $\Delta(t) \equiv 0$, or $\Delta(t) < 0$ everywhere on Γ.

Theorem 6. Let $\varkappa = \operatorname{Ind}_\Gamma G(t)$. All assertions of Theorem 1 are valid for problem (6) in the following cases:

1) $a_1, b_1, a_2, b_2, c_1, c_2$ satisfy the Hölder condition and the boundary conditions are of elliptic type;

2) a_1, b_1, a_2, b_2 are continuous, $c_1, c_2 \in L_p(\Gamma)$, $p > 1$, and the condition of strengthened ellipticity is satisfied

$$\frac{\Delta(t)}{|G(t)|^2} > 1 - \frac{2}{1 + \mathfrak{M}_p}.$$

Remark. In the parabolic case problem (6) reduces to two related problems with an oblique derivative (7).

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Note: Figure translations are in progress. See original paper for figures.