THEORY OF ELASTICITY

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Fig. 1

Figure 1: Fig. 1

Abstract

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THEORY OF ELASTICITY

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ON THE STABILITY OF A THICK RECTANGULAR SIMPLY SUPPORTED PLATE UNDER A COMPRESSIVE LOAD

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The stressed and deformed state of thick plates was studied by V. G. Galerkin (1), who proceeded from the general equations of the theory of elasticity. In the present work the stability of a thick rectangular simply supported plate under the action of a uniformly distributed compressive load applied to two opposite sides is considered. The loss of stability of sufficiently thick plates is accompanied by residual deformations; therefore, below we use the relations of the theory of small elastic-plastic deformations (2). Following the ideas of L. S. Leibenzon (3) and A. Yu. Ishlinskii (4), the study of the process of loss of stability is carried out from the general relations of the laws connecting stresses and deformations, without resorting to Kirchhoff–Love hypotheses.

Consider a rectangular thick simply supported plate under the action of a uniformly distributed compressive load directed along the z-axis. The dimensions are indicated in Fig. 1. Neglecting, for simplicity, the compressibility of the material, we write the relations of the theory of small elastic-plastic deformations in the form

Fig. 1

$$\sigma_x - \sigma = \frac{2}{3}\frac{\sigma_i}{e_i} e_x, \dots, \qquad \tau_{xy} = \frac{1}{3}\frac{\sigma_i}{e_i} e_{xy}, \dots,$$

$$\sigma_i = \Phi(e_i), \qquad \sigma = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z), \tag{1}$$

$$\sigma_i = \frac{\sqrt{2}}{2}\sqrt{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)},$$

$$e_i = \frac{\sqrt{3}}{2}\sqrt{(e_x - e_y)^2 + (e_y - e_z)^2 + (e_z - e_x)^2 + \frac{3}{2}(e_{xy}^2 + e_{xz}^2 + e_{yz}^2)},$$

where σ_{ij}, e_{ij} are, respectively, the components of stress and deformation.

Denoting by u, v, w the displacement components along the axes x, y, z, we shall seek the solution in the form

$$\sigma_{ij} = \sigma_{ij}^0 + \sigma_{ij}'; \qquad e_{ij} = e_{ij}^0 + e_{ij}'; \qquad u = u^0 + u', \dots, \tag{2}$$

where the superscript zero is assigned to the components of the unperturbed state, and the superscript prime to the components of the perturbation.

It is easy to see that

$$\sigma_z^0 = \sigma_i^0 = -p, \qquad e_x^0 = e_y^0 = -\frac{1}{2}e_z^0, \qquad e_i = e_z^0, \qquad \Phi(e_z^0) = -p, \tag{3}$$

$$\sigma_x^0 = \sigma_y^0 = \tau_{xy}^0 = \tau_{xz}^0 = e_{xy}^0 = e_{xz}^0 = e_{yz}^0 = 0.$$

Substituting expression (2) into relations (1), linearizing them and taking (3) into account, we obtain

$$\begin{gathered}
\sigma_x' - \sigma' = -\frac{p}{3e_z^0}(e_x' - e_y') - \frac{1}{3}\left[\sigma_z' - \frac{1}{2}(\sigma_x' + \sigma_y')\right], \\
\sigma_y' - \sigma' = -\frac{p}{3e_z^0}(e_y' - e_x') - \frac{1}{3}\left[\sigma_z' - \frac{1}{2}(\sigma_x' + \sigma_y')\right], \\
\sigma_z' - \sigma' = \frac{2}{3}\left[\sigma_z' - \frac{1}{2}(\sigma_x' + \sigma_y')\right], \\
\tau_{yx}' = -\frac{p}{3e_z^0}e_{xy}', \qquad \tau_{zx}' = -\frac{p}{3e_z^0}e_{zx}', \qquad \tau_{yz}' = -\frac{p}{3e_z^0}e_{yz}', \\
\sigma_i = \left(\frac{d\Phi}{de_i}\right)^0 e_i', \qquad \sigma_i' = \sigma_z' - \frac{1}{2}(\sigma_x' + \sigma_y'), \qquad e_i' = \frac{2}{3}\left[e_z' - \frac{1}{2}(e_y' + e_x')\right],
\end{gathered} \tag{4}$$

where $(d\Phi/de_i)^0$ means that the derivative is taken at $e_i = e_i^0$. Obviously, the components of the perturbation must satisfy the equilibrium equations

$$\frac{\partial \tau_x'}{\partial x} + \frac{\partial \tau_{xy}'}{\partial y} + \frac{\partial \tau_{xz}'}{\partial z} = 0, \dots \tag{5}$$

From (4) and (5) it follows that

$$\frac{\partial \sigma_x'}{\partial x} = B_0 \left[\Delta u' - 2\frac{\partial^2 u'}{\partial x^2} \right], \qquad \frac{\partial \sigma_y'}{\partial y} = B_0 \left[\Delta v' - 2\frac{\partial^2 v'}{\partial y^2} \right], \tag{6}$$

$$\frac{\partial \sigma_z'}{\partial z} = B_0 \left[\Delta w' - 2\frac{\partial^2 w'}{\partial z^2} \right], \qquad B_0 = \frac{p}{3e_z^0}, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Using further the incompressibility condition, the law relating stress and strain intensities, and also the remaining relations (4), we obtain the initial system of equations

$$(6+k) \left[\frac{\partial^5 u'}{\partial x^2 \partial y\, \partial z^2} + \frac{\partial^5 v'}{\partial x \partial y^2 \partial z^2} \right] - 2 \left[\frac{\partial^3 \Delta u'}{\partial x^2 \partial y} + \frac{\partial^3 \Delta v'}{\partial x \partial y^2} \right] - \frac{\partial^2}{\partial z^2} \left[\frac{\partial \Delta u'}{\partial y} + \frac{\partial \Delta v'}{\partial x} \right] = 0, \tag{7}$$

$$\frac{\partial \Delta u'}{\partial y} - \frac{\partial \Delta v'}{\partial x} = 0,$$

where

$$k = 6\frac{e_z^0}{p} \left(\frac{d\Phi}{de_i} \right)^0 = -6\frac{E_k}{E_c},$$

E_k is the tangential modulus, E_c the secant modulus. It is easy to see that $-6 \leq k \leq 0$. We shall seek the solution in the form

$$u' = F(x) \cos(ny) \cos(mz), \qquad v' = G(x) \sin(ny) \sin(mz), \tag{8}$$

i.e., restricting ourselves to consideration of buckling symmetric with respect to the y, z axes. From (8), (4), and (6) we find

$$\sigma_x' = -B_0 \left[\frac{dF}{dx} + (m^2 + n^2) \int F\, dx \right] \cos(ny) \cos(mz),$$

$$\sigma_y' = -\frac{B_0}{n} \left[\frac{d^2 G}{dx^2} + (n^2 - m^2) G \right] \cos(ny) \cos(mz),$$

$$\sigma_z' = -\frac{B_0}{m^2} \left[(n^2 - m^2) \left(\frac{dF}{dx} + nG \right) - \left(\frac{d^3 F}{dx^3} + n\frac{d^2 G}{dx^2} \right) \right] \cos(ny) \cos(mz),$$

$$\tau'_{xy} = B_0 \left[nF - \frac{dG}{dx} \right] \sin(ny)\cos(mz),$$

$$\tau'_{yz} = -\frac{B_0}{m} \left[n\frac{dF}{dx} + (n^2 - m^2)G \right] \sin(ny)\sin(mz), \tag{9}$$

$$\tau'_{zx} = \frac{B_0}{m} \left[\frac{d^2F}{dx^2} + m^2F + n\frac{dG}{dx} \right] \cos(ny)\sin(mz),$$

$$w' = -\frac{1}{m} \left[\frac{dF}{dx} + nG \right] \cos(ny)\sin(mz).$$

In determining expressions (9), arbitrary constants have been omitted; this is connected with the satisfaction of the boundary conditions.

For a freely supported plate we must have $u' = 0$ for $y = \pm b$, $z = \pm l$. Then from (8) we find

$$nb = \pm\pi/2 + i\pi, \qquad ml = \pm\pi/2 + j\pi, \qquad i, j = 0, 1, 2, \ldots \tag{10}$$

In this case, from (9) we obtain

$$\sigma'_y = 0 \quad \text{for } y = \pm b; \qquad \sigma'_z = 0 \quad \text{for } z = \pm l. \tag{11}$$

Thus, when the plate buckles, its lateral faces $y = \pm b$, $z = \pm l$ are free of bending and twisting moments. Substituting expression (8) into equation (7), we find

$$\begin{gathered}
2n\left[\frac{d^4F}{dx^4} - (m^2+n^2)\frac{d^2F}{dx^2}\right] - nm^2\left[\frac{d^2F}{dx^2} - (m^2+n^2)F\right] + \\
+ (2n^2+m^2)\left[\frac{d^3G}{dx^3} - (m^2+n^2)\frac{dG}{dx}\right] + (6+k)nm^2\left[\frac{d^2F}{dx^2} + n\frac{dG}{dx}\right] = 0,
\end{gathered} \tag{12}$$

$$n\left[\frac{d^2F}{dx^2} - (m^2+n^2)F\right] + \left[\frac{d^3G}{dx^3} - (m^2+n^2)\frac{dG}{dx}\right] = 0.$$

The function $F(x)$, determined from system (12), is represented as a sum of even and odd functions. The odd functions correspond to loss of stability symmetric with respect to the z-axis (formation of a neck); the even functions correspond to lateral buckling of the plate.

Restricting ourselves to consideration of lateral buckling of the plate, we obtain

$$F = C_1 \operatorname{ch} \lambda x + C_2 \cos \gamma x + C_3 A \operatorname{ch} \sigma x,$$
$$G = -\frac{nC_1}{\lambda} \operatorname{sh} \lambda x - \frac{nC_2}{\gamma} \sin \gamma x + \frac{C_3}{\sigma}(1 - nA) \operatorname{ch} \sigma x, \tag{13}$$

where C_1, C_2, C_3 are arbitrary constants; $\sigma^2 = m^2 + n^2$; $A = -n/m^2$;

$$\lambda = \frac{1}{2}\left\{-\left[(6+k)m^2 - 4\sigma^2\right] + \left\{\left[(6+k)m^2 - 4\sigma^2\right]^2 - 8\left[2\sigma^4 - (6+k)n^2m^2\right]\right\}^{1/2}\right\}^{1/2},$$

$$\gamma = \frac{1}{2}\left\{\left[(6+k)m^2 - 4\sigma^2\right] + \left\{\left[(6+k)m^2 - 4\sigma^2\right]^2 - 8\left[2\sigma^4 - (6+k)n^2m^2\right]\right\}^{1/2}\right\}^{1/2}.$$

Let us proceed to the consideration of the boundary conditions on the lateral surface. When buckling occurs, the points of the lateral surface receive displacements u', v', w'. If the equation of the buckled lateral surface is written in the form

$$x = \pm a + f(\pm a, y, z), \tag{14}$$

then the coordinates of the point $M(\pm a + u',\ y + v',\ z + w')$ must satisfy equation (14). Hence, linearizing, we obtain

$$u'(\pm a, y, z) = f(\pm a, y, z). \tag{15}$$

The lateral surface is free of stresses; consequently, on it

$$\sigma'_x \alpha_1 + \tau'_{xy}\alpha_2 + \tau'_{xz}\alpha_3 = 0, \ldots, \tag{16}$$

where $\alpha_1, \alpha_2, \alpha_3$ are the direction cosines of the normal to the lateral surface of the plate. It is easy to obtain, up to infinitesimals of the second order,

$$\alpha_1 \approx 1, \qquad \alpha_2 = -\partial u'/\partial y, \qquad \alpha_3 = -\partial u'/\partial z. \tag{17}$$

Using (17) and (16), and linearizing, we finally find

$$\sigma'_x = 0, \qquad \tau'_{xy} = 0, \qquad \tau'_{xz} + p\,\partial u'/\partial z = 0 \quad \text{for } x = \pm a. \tag{18}$$

Substituting expressions (8) and (9) into (18), and taking (13) into account, we obtain a homogeneous linear system of three equations with respect to three unknowns

Fig. 2

Figure 2: Fig. 2

C_1, C_2, C_3. Equating the determinant to zero, we obtain the transcendental equation for determining the critical deformation

$$\begin{aligned} e_z^0 = \{&\gamma(\sigma^2+\lambda^2)[m^2(\sigma^2-\gamma^2)-2n^2(\sigma^2+\gamma^2)]\,\mathrm{th}(\lambda a) \\ &-\lambda(\sigma^2-\gamma^2)[m^2(\sigma^2+\lambda^2)-2n^2(\sigma^2-\lambda^2)]\,\mathrm{tg}(\gamma a) \\ &+4n^2\sigma\gamma\lambda(\gamma^2+\lambda^2)\,\mathrm{th}(\sigma a)\}/\{3m^4[\gamma(\sigma^2+\lambda^2)\,\mathrm{th}(\lambda a)-\lambda(\sigma^2-\gamma^2)\,\mathrm{tg}(\gamma a)]\}. \end{aligned} \tag{19}$$

The following method may be proposed for solving equation (19). For given parameters m, n, a, one should prescribe the value k; then equation (19) determines the critical value of the deformation e_z^{0*}. By prescribing the secant modulus E_c, we determine the point D (Fig. 2) at which the tangent modulus $E_k = -kE_c/6$ will be determined. Thus, to each point of the plane p^*, e_z^{0*} there may be put in correspondence the critical values E_k^*, E_c^*.

The critical force is determined as follows: for a given curve of uniaxial compression $p = \Phi(e_z^0)$, the points are determined at which the tangent and secant moduli coincide with the critical values E_k^*, E_c^*. In this way the values p^*, e_z^{0*} are determined. In the case of determining several critical-force values, the smallest is the most dangerous.

Fig. 2

For small thicknesses, formula (19) can be simplified:

$$e_z^0 = -\frac{a^2}{9m^4}\{m^2(\sigma^2-\gamma^2)(\sigma^2+\lambda^2)+2n^2(\sigma^2+\gamma^2)(\sigma^2-\lambda^2)\}. \tag{20}$$

If the loss of stability is not accompanied by passage beyond the yield limit, then the displacement components are biharmonic, and the basic equation (12) for determining the function F takes the form

$$\frac{d^4F}{dx^4} - 2\sigma^2\frac{d^2F}{dx^2} + \sigma^4 F = 0. \tag{21}$$

Carrying out analogous reasoning, we obtain the equation for determining the critical deformation e_z^{0*} in the form

$$e_z^{0*} = \frac{1}{3m^2a}\,[2\sigma^2 a - \sigma\,\mathrm{sh}(2\sigma a)]. \tag{22}$$

For small a, expanding $\mathrm{sh}(2\sigma a)$ in a series and retaining the second term of the expansion, we obtain

$$e_z^{0*} = \sigma^4 (2a)^2 / 9m^2. \tag{23}$$

Assuming that in the direction of the y-axis the number of half-waves is equal to unity, and denoting by v the number of half-waves along the z-axis, we transform formula (23) to the form

$$e_z^{0*} = \frac{\pi^2 D}{2Ea(2l)^2} \left[\nu + \frac{1}{\nu} \frac{l^2}{b^2} \right]^2, \qquad D = \frac{E(2a)^3}{9}. \tag{24}$$

Formula (24) coincides exactly with the formula for the critical deformation according to the Kirchhoff–Love plate theory (^{5}).

Thus, the formulas of plate theory are limiting cases of the relations obtained.

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CITED LITERATURE

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