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Abstract

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MATHEMATICS

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ON THE METHOD OF AVERAGING

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§ 1. Formulation of the problem.

Consider the system

$$\dot{x} = \varepsilon X(x, y, t, \varepsilon), \qquad \dot{y} = Y(x, y, t, \varepsilon); \tag{1}$$

x, X and y, Y are, respectively, n- and m-dimensional vectors; $\varepsilon > 0$ is a small parameter. For $\varepsilon = 0$, (1) turns into the degenerate system

$$\dot{y} = Y(x, y, t, 0) \equiv Y_0(x, y, t), \qquad x = \text{const}. \tag{2}$$

The general solution of (2) is assumed known:

$$y = \varphi(x, y_0, t_0, t) \quad \left(\varphi(x, y_0, t_0, t_0) \equiv y_0, \quad \operatorname{rank}\left\|\frac{\partial \varphi}{\partial y_0}, \frac{\partial \varphi}{\partial t_0}\right\| = m\right). \tag{3}$$

Suppose that along every solution of (2) the right-hand sides of (1) and the other functions that occur have mean values independent of y_0, t_0 (in § 4 it is shown that the assumption of independence of y_0, t_0 entails no loss of generality). We form the averaged system

$$\dot{\bar{x}} = \varepsilon \bar{X}_1(\bar{x}) \equiv \lim_{T \to \infty} \frac{1}{T_0} \int_{t_0}^{t_0+T} X[\bar{x}, \varphi(\bar{x}, y_0, t_0, t), t, 0]\, dt. \tag{4}$$

Problem. Compare the solutions of (1) and (4) on a large interval $t \sim 1/\varepsilon$, and construct averaged systems of higher orders

$$\begin{aligned} \dot{\bar{x}} &= \varepsilon \bar{X}_1(\bar{x}) + \varepsilon^2 A_2(\bar{x}) + \varepsilon^3 \ldots, \\ \dot{\bar{y}} &= Y_0(\bar{x}, \bar{y}, t) + \varepsilon B_1(\bar{x}) + \varepsilon^2 B_2(\bar{x}) + \varepsilon^3 \ldots, \end{aligned} \tag{5}$$

which approximate the solutions of (1) with greater accuracy depending on the number of retained terms.

§ 2. Formal expansions.

Write (1) in the form

$$\begin{aligned} \dot{x} &= \varepsilon X_1(x,y,t) + \varepsilon^2 X_2(x,y,t) + \varepsilon^3 \dots, \\ \dot{y} &= Y_0(x,y,t) + \varepsilon Y_1(x,y,t) + \varepsilon^2 Y_2(x,y,t) + \varepsilon^3 \dots . \end{aligned} \tag{6}$$

For (6) we seek a transformation

$$\begin{aligned} x &= \bar{x} + \varepsilon u_1(\bar{x},\bar{y},t) + \varepsilon^2 u_2(\bar{x},\bar{y},t) + \varepsilon^3 \dots, \\ y &= \bar{y} + \varepsilon v_1(\bar{x},\bar{y},t) + \varepsilon^2 v_2(\bar{x},\bar{y},t) + \varepsilon^3 \dots, \end{aligned} \tag{7}$$

leading to (5). Differentiating (7), using (5), and equating in (6) the coefficients of powers of ε, we obtain an infinite system of equalities for the terms of (5), (7). Their choice is not unique; the functions $A_2, A_3, \dots, B_1, B_2, \dots$ can, in general, be chosen arbitrarily.

For $u_1, v_1, u_2, v_2, \dots$ one obtains equations of the form

$$\frac{\partial u_1}{\partial t} + \left(Y_0 \frac{\partial}{\partial y}\right) u_1 = X_1 - \overline{X}_1 \equiv S, \tag{8}$$

$$\frac{\partial v_1}{\partial t} + \left(Y_0 \frac{\partial}{\partial y}\right) v_1 - \left(v_1 \frac{\partial}{\partial y}\right) Y_0 = R \tag{9}$$

(R is a certain known function, $\partial/\partial x = \{\partial/\partial x_1, \dots, \partial/\partial x_n\}$, $\partial/\partial y = \{\partial/\partial y_1, \dots, \partial/\partial y_m\}$), which are easily solved successively, since the solutions of the characteristic systems are known: for (8), $y = \varphi(x, y_0, t_0, t)$,

$$u = u_0 + \int_{t_0}^{t} S\,dt;$$

for (9), $y = \varphi$, $v = z(v_0, x_0, y_0, t_0, t)$, where z is the general solution of the linear system $\dot{z} = (z\,\partial/\partial y)Y_0 + R$, expressed in terms of the fundamental solutions of the homogeneous system $\dot{z} = (z\,\partial/\partial y)Y_0$, which are the elements of the matrix $\|\partial\varphi/\partial y_0,\ \partial\varphi/\partial t_0\|$. Thus, the formal expansions (5), (7) are found to arbitrary accuracy.

§ 3. **Justification of the method.** The domain of definition of (1) is $0 \le \varepsilon \le \varepsilon_0$; $x, y, t \in G$; G is an open domain of the space x, y, t. We compare the solutions $x = x(x_0, y_0, t_0, t, \varepsilon)$, $y = y(x_0, y_0, t_0, t, \varepsilon)$ of system (1) and $\bar{x} = \bar{x}(x_0, t_0, t, \varepsilon)$ of system (4) with common fixed initial point $x_0, y_0, t_0 \in G$.

Let:

1) X, Y be continuous in ε uniformly with respect to x, y, t, ε.

2) $X_1 \equiv X|_{\varepsilon=0}$, $Y_0 \equiv Y|_{\varepsilon=0}$ be continuous and satisfy a Lipschitz condition in x; X_1 is continuously differentiable in y; $|\partial X/\partial y| \leq \text{const} < \infty$.

3) Through each point of G there passes a unique integral curve (3) of system (2), lying in G for $t_0 \leq t < \infty$, and extendable for $t \leq t_0$ up to the boundary of G or to $t > -\infty$.

4) The function (3) is continuous and continuously differentiable with respect to y_0, t_0;

$$|\partial\varphi/\partial y_0|,\ |\partial\varphi/\partial t_0| \leq \text{const} < \infty; \qquad \sum_{i=1}^{m+1} D_i^2 \geq \sigma > 0;$$

D_i are the minors of order m of the matrix $\|\partial\varphi/\partial y_0,\ \partial\varphi/\partial t_0\|$.

5) In G there lies a manifold M, given parametrically: $x = a(\lambda)$, $y = b(\lambda)$, $t = c(\lambda)$ ($\lambda = \{\lambda_1, \dots, \lambda_{n+m}\} \in \Lambda$, Λ an open domain). The functions $a(\lambda)$, $b(\lambda)$, $c(\lambda)$ are continuous and continuously differentiable;

$$|\partial a/\partial\lambda|,\ |\partial b/\partial\lambda|,\ |\partial c/\partial\lambda| \leq \text{const} < \infty, \qquad \sum_{i=1}^{n+m+1} A_i^2 \geq \sigma > 0;$$

A_i are the minors of order $n+m$ of the matrix $\|\partial a/\partial\lambda,\ \partial b/\partial\lambda,\ \partial c/\partial\lambda\|$.

6) The absolute values of the angles of intersection of the curves (3) with M are bounded below by a positive number.

7) In G every curve (3) intersects M. If the point of intersection is not unique, then the projections of the curve (3) and M in the space x, y intersect no more than once, in a neighborhood of the curve $\partial X_1/\partial t \equiv \partial Y_0/\partial t \equiv 0$, $|Y_0| \neq 0$.

8) Uniformly with respect to $x_0, y_0, t_0 \in G$, there exists the limit (4). $\overline{X}_1$ is bounded, satisfies a Lipschitz condition, $|X_1 - \overline{X}_1| \leq \text{const} < \infty$; the derivatives of

$$\int_{t_0}^{t_0+T} X_1(x, \varphi, t)\, dt$$

with respect to y_0, t_0 ($0 \leq T < \infty$) are uniformly bounded.

9) For $0 < \varepsilon \leq \varepsilon_0$ there exist open bounded subdomains $G_0(\varepsilon) \subseteq G$, containing x_0, y_0, t_0 together with some ρ-neighborhood ($\rho = \text{const} > 0$). The transition time of the curve (3) from any point of G_0 to the nearest, in time, point of intersection with M does not exceed in absolute value K/ε ($K = \text{const} > 0$).

10) For $0 < \varepsilon \leqslant \varepsilon_0$ there exist open subdomains $G_1(\varepsilon) \subset G_0$, containing x_0, y_0, t_0; the distances from the points of G_1 to the boundary of G_0 are bounded below by a positive number. Introduce the interval $[t_0, t_1(\varepsilon)]$: $t_1 > t_0$, $t_1 - t_0 \leqslant K/\varepsilon$; for $t_0 \leqslant t \leqslant t_1$ the solutions $x(x_0, y_0, t_0, t, \varepsilon)$, $y(x_0, y_0, t_0, t, \varepsilon)$ do not leave G_1.

Theorem. For any $K > 0$, $\delta > 0$ there exists an $\varepsilon_1 > 0$ ($\varepsilon_1 \leqslant \varepsilon_0$) such that, for $0 < \varepsilon \leqslant \varepsilon_1$, $t \in [t_0, t_1(\varepsilon)]$, $\bar{x}$ does not leave G and $|x - \bar{x}| \leqslant \delta$.

Remark 1. If one assumes that in G as a whole there exists a solution of (8), continuous in x, having continuous bounded derivatives with respect to y, t, then conditions 4), 5), 6), 7), 8) may be dispensed with, assuming only the existence of the uniform limit (4), boundedness, and the Lipschitz condition for $\bar{X}_1$.

Remark 2. If one requires that the solution of (8) be bounded in G together with $\partial u/\partial x$, then one may assert that $|x - \bar{x}| = O(\varepsilon)$ for $t \sim 1/\varepsilon$.

The theorems on higher approximations of the method of §§ 1 and 2 are formulated and proved analogously. An analogue of the theorem from (^{1}) has also been proved: namely, under certain conditions, in a neighborhood of an equilibrium point of system (4) there exists a solution of (1) that attracts or repels nearby solutions as $t \to \pm\infty$. These theorems are not given here because of the cumbersomeness of their formulations.

§ 4. **More general systems.** Let a system of the type (1), (6) satisfy the conditions of §§ 1-3; let the mean values of the functions occurring exist, but be allowed to depend on y_0, t_0. We describe this as follows: the general solution (2) has the form $y = \varphi(c, t)$ ($c = \{c_1, \dots, c_m\}$ are arbitrary constants); the mean values depend on $c_1, \dots, c_k$ and do not depend on $c_{k+1}, \dots, c_m$ ($k \leqslant m$); to the constants $c_1, \dots, c_k$ there correspond the integrals

$$c = \Phi(x, y, t) \qquad (c = \{c_1, \dots, c_k\},\ \Phi = \{\Phi_1, \dots, \Phi_k\}), \tag{10}$$

on the integral surfaces (10) the solutions of (2) admit the parametric representation

$$y = \psi(c_1, \dots, c_k, z, x, t) \tag{11}$$

($z = \{z_1, \dots, z_{m-k}\}$ is the set of parameters). (2) and (11) generate on the surfaces (10) the system $\dot{z} = Z(z, c, t, x) \equiv (Y_0 \partial/\partial y + \partial/\partial t)\theta$, where $\theta = \theta(x, y, t)$, $\psi[\Phi(x, y, t), \theta(x, y, t), x, t] \equiv y$, with general solution $z = z(z_0, t_0, c, x, t)$. Introduce in (1), (6) the new variables $x = x$, $c = \Phi(x, y, t)$, $z = \theta(x, y, t)$; we obtain the system

$$\begin{aligned}\dot{x} &= \varepsilon X \equiv \varepsilon P(x, c, z, t, \varepsilon),\\ \dot{c} &= \varepsilon \left(X \frac{\partial}{\partial x} \right) \Phi \equiv \varepsilon Q(x, c, z, t, \varepsilon),\\ \dot{z} &= Z + \varepsilon \left(X \frac{\partial}{\partial x} \right) \theta \equiv L(x, c, z, t, \varepsilon).\end{aligned} \tag{1}$$

(12) belongs to type (1); the mean values do not depend on z_0, t_0, and therefore the theory of §§ 1-3 is applicable. In this sense it was stated in § 1 that the assumption of independence of the mean values from y_0, t_0 does not restrict generality.

§ 5. **Perturbed systems with slowly varying parameters.** One may regard (2) as the unperturbed system, and (6) as a system perturbed by the functions $\varepsilon Y_1 + \varepsilon^2 Y_2 + \cdots$, containing slowly varying parameters x. One seeks a representation of the solutions of (6) in terms of the solutions of (2). Under the conditions of § 4, averaging (12) by the method of §§ 1-3 and computing approximations for x, c, z, we find a representation of the solutions of (6) in the form $y = \psi(c, z, x, t)$ with any desired degree of accuracy. For systems of a special form this problem in the first approximation was consid-

was considered in ($^{3-5}$); in (^{5}) a similar problem was studied for the case when the surfaces (10) are bounded and closed, and the system (2) has an integral invariant. In (^{6}) an asymptotic method is proposed, connected with an averaging scheme different from §§ 1-3 and having another domain of application.

§ 6. **Canonical systems.** Let the unperturbed system

$$\dot{q} = \frac{\partial}{\partial p} H(p, q, x), \qquad \dot{p} = -\frac{\partial}{\partial q} H(p, q, x), \qquad x = \text{const}, \tag{13}$$

correspond to the perturbed one:

$$\dot{q} = \frac{\partial}{\partial p} H + \varepsilon f^{(p)}(p, q, x, t, \varepsilon), \qquad \dot{p} = -\frac{\partial}{\partial q} H + \varepsilon f^{(q)}(p, q, x, t, \varepsilon),$$

$$\dot{x} = \varepsilon X(p, q, x, t, \varepsilon). \tag{14}$$

For (13) there exists an energy integral $E = H(p, q, x)$. If (13), (14) satisfy the conditions of § 5, then for E we find:

$$\dot{\bar{E}} = \varepsilon \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0 + T} \left(f^{(p)}(p, q, x, t, 0) \dot{p} + f^{(q)}(p, q, x, t, 0) \dot{q} + \frac{\partial H}{\partial x} X \right) dt. \tag{15}$$

According to (15), the rate of change of E is equal to the mean power of the forces $\varepsilon f^{(p)}$, $\varepsilon f^{(q)}$, εX. For systems of a special form, (15) was derived in (^{3,4}).

§ 7. **Special cases.** Special cases of (1) are systems in standard form and systems with a rapidly rotating phase, studied in (^{1,2}).

Summary. An averaging method has been developed and justified for systems of the general form (1), which is a generalization of the methods (^{1,2}).

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