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Abstract

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ON PERIODIC SOLUTIONS OF BOUNDARY-VALUE PROBLEMS WITHOUT INITIAL CONDITIONS FOR PARABOLIC EQUATIONS

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Let D be a bounded m-dimensional domain in the space of variables $(x_1, \dots, x_m) = x$, and let Γ be the boundary of D. Denote by $Q = D \times (-\infty, +\infty)$ the straight cylinder of $m+1$ dimensions and by S the lateral surface of Q. The part of Q enclosed between the planes $t = t_0$ and $t = t_0 + T$, where t_0 is any number in $(-\infty, +\infty)$, and T is a fixed number, will be denoted by Q_T, and the lateral surface of Q_T by S_T. By L we shall denote the elliptic operator:

$$L \equiv \sum_{i,j=1}^{m} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_i(x) \frac{\partial}{\partial x_i} + c(x),$$

where

$$\sum_{i,j=1}^{m} a_{ij}(x)\xi_i\xi_j \geq \alpha \sum_{i=1}^{m} \xi_i^2 \quad (\alpha = \text{const} > 0) \quad \text{and} \quad a_{ij}(x) = a_{ji}(x).$$

Theorem 1. Let $\overline{D}$ belong to the class $A^{(1,\lambda)}$ (1), and let the coefficients of L satisfy in $\overline{D}$ the conditions:

$$a_{ij}(x) \in C^{(1,\lambda)} \text{ (1)}, \quad b_i(x) \in C^{(0,\lambda)}, \quad c(x) \in C^{(0,\lambda)},$$

$$-c(x) \geq c_0 \quad (c_0 = \text{const} > 0).$$

Suppose, in addition, that the functions $f(x,t)$, $\varphi(x,t)$, periodic in t with period T, satisfy the conditions: $f(x,t)$ is continuous in Q together with its derivatives with respect to t up to the 4th order inclusive and is Hölder continuous in $x \in D$ with exponent λ, while $\varphi(x,t)$ is continuous on S with derivatives with respect to t up to the 4th order inclusive. Under the stated conditions there exists a unique regular* solution, periodic in t with period T, of the boundary-value problem

$$u_t = Lu + f(x,t); \tag{1}$$

$$u|_S = \varphi(x,t). \tag{2}$$

Proof. The periodic solution of problem (1), (2) is sought in the form of the series

$$u(x,t) = X_0(x)d_0 + \sum_{k=1}^{\infty}(X_k(x)d_k(t) + Y_k(x)e_k(t)), \tag{3}$$

where

$$d_0 = \sqrt{\frac{1}{T}}, \qquad d_k(t) = \sqrt{\frac{2}{T}}\cos\frac{2\pi k}{T}t, \qquad e_k(t) = \sqrt{\frac{2}{T}}\sin\frac{2\pi k}{T}t$$

$$(k = 1, 2, ...). \tag{4}$$

Substituting the series (4) into equation (1) and the boundary condition (2), and using the orthogonality of the system (4), for determining $X_0(x)$, $X_k(x)$, and $Y_k(x)$ we obtain a sequence of boundary-value problems:

$$LX_0 + f_0(x) = 0, \qquad X_0|_\Gamma = \varphi_0(x); \tag{5}$$

$$LX_k + c_{kY}k + \tilde{f}_k(x) = 0, \qquad X_k|_\Gamma = \widetilde{\varphi}_k(x),$$

$$LY_k + c_{kX}k + \tilde{\tilde{f}}_k(x) = 0, \qquad Y_k|_\Gamma = \widetilde{\widetilde{\varphi}}_k(x) \quad (k = 1, 2, ...), \tag{6}$$

where $f_0(x), \tilde{f}_k(x), \tilde{\tilde{f}}_k(x)$ and $\varphi_0(x), \widetilde{\varphi}_k(x), \widetilde{\widetilde{\varphi}}_k(x)$ are the Fourier coefficients of the functions $f(x,t)$ and $\varphi(x,t)$, respectively, with respect to the system (4), and $c_k = 2\pi k/T$.

Passing to the analysis of the boundary-value problems (5) and (6), we shall use the results set forth in ([1]). Problem (5) is a Dirichlet problem having a

[*] That is, continuous in $\overline{Q}$ and possessing inside Q continuous derivatives that enter equation (1).

unique solution regular in D. Fixing k, let us consider problem (6). Following the idea of the work (^{2}) and estimating, by the maximum principle, the square of the length $X_k^2 + Y_k^2$ of the solution of problem (6) regular in D, we obtain the estimate in $\overline{D}$:

$$X_k^2 + Y_k^2 \le \sup\left\{\max_{x\in\Gamma}\left(\widetilde{\varphi}_k^2 + \widetilde{\widetilde{\varphi}}_k^2\right), \frac{1}{c_0}\max_{x\in\overline{D}}\left(\tilde{f}_k^2 + \tilde{\tilde{f}}_k^2\right)\right\}, \tag{7}$$

which ensures uniqueness of the solution of problem (6) regular in D.

Consider the system of integral equations:

$$X_k(x) = -c_k\int_D F(x,y)Y_k(y)\,dy + \int_D F(x,y)\tilde{f}_k(y)\,dy - \int_\Gamma P_{yF}(x,y)\widetilde{\varphi}_k(y)\,d_y\sigma,$$

$$Y_k(x) = c_k\int_D F(x,y)X_k(y)\,dy + \int_D F(x,y)\tilde{\tilde{f}}_k(y)\,dy - \int_\Gamma P_{yF}(x,y)\widetilde{\widetilde{\varphi}}_k(y)\,d_y\sigma, \tag{8}$$

where $F(x,y)$ is the Green’ s function of the first boundary-value problem for the operator L and the domain D; P_y is a linear differential operator of first order.

Every continuous in $\overline{D}$ solution of the system (8) is also a regular solution of problem (6), and, consequently, the homogeneous system of integral equations corresponding to system (8) has only the trivial solution. But then, on the basis of the theory of equations with completely continuous operators (^{3}), system (8) has a unique solution continuous in $\overline{D}$, i.e. the problem is regularly solvable in D.

Estimate (7) and the properties of the functions $f(x,t)$ and $\varphi(x,t)$ ensure uniform convergence in $\overline{Q}$ both of the series (3) and of the series obtained from (3) by termwise differentiation with respect to t twice.

The proof that the function $u(x,t)$ represented by the series (3) has inside Q derivatives with respect to x up to order 2 inclusive and satisfies equation (1) inside Q is carried out in the same way as in the monograph (^{4}) (p. 744). The uniqueness of the solution of the problem under consideration follows from the periodicity of the solution and the maximum principle.

A theorem analogous to Theorem 1 also holds in the case when D is an unbounded domain complementing some bounded domain to the whole space, and $f(x,t)$ has a definite order of decrease as $x_1^2 + \dots + x_m^2 \to \infty$.

Let us pass to the question of periodic solutions of the first boundary-value problem:

$$\frac{\partial u}{\partial t} - \sum_{i,j=1}^{m} \frac{\partial}{\partial x_i}\left(a_{ij}(x,t)\frac{\partial u}{\partial x_j}\right) + c(x,t)u - f(x,t) = 0, \tag{9}$$

$$u|_S = 0, \tag{10}$$

where

$$\sum_{i,j=1}^{m} a_{ij}(x,t)\xi_i\xi_j \geq a\sum_{i=1}^{m}\xi_i^2 \quad (a = \text{const} > 0)$$

and $a_{ij}(x,t) = a_{ji}(x,t)$.

Definition. We shall call a function $u(x,t)$ a **weak periodic** solution with period T of problem (9), (10), if it satisfies the following requirements: 1) if $t_0 \in (-\infty, +\infty)$, then

$$u(x, t_0 + T) = u(x, t_0) \tag{11}$$

for almost all $x \in D$; 2) $u(x,t) \in \overset{0}{W}{}_2^1(Q_T)$; 3) for every function $\Phi(x,t) \in \overset{0}{W}{}_2^1(Q_T)$, periodic in t with period T, the function $u(x,t)$ satisfies the integral identity

$$\int_{Q_T}\left(\frac{\partial u}{\partial t}\Phi + \sum_{i,j=1}^{m} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial \Phi}{\partial x_j} + cu\Phi - f\Phi\right) dt\,dx = 0. \tag{12}$$

Theorem 2. *Let the boundary Γ of the domain D be $m+3$ times continuously differentiable, and let the coefficients of equation (9) and the function $f(x,t)$ be periodic in t with period T and possess in $\bar{Q}$ the following properties: 1) $a_{ij}(x,t)$ are continuous together with derivatives of the form $\partial a_{ij}/\partial t$, $\partial^k a_{ij}/\partial x_1^{k_1} \ldots \partial x_m^{k_m}$ $(k = 1, \ldots, m+2)$; 2) $c(x,t)$ is continuous together with derivatives of the form $\partial c/\partial t$, $\partial^k c/\partial x_1^{k_1} \ldots \partial x_m^{k_m}$ $(k = 1, \ldots, m)$, and $c(x,t) \geq c_0$, where $c_0 = \text{const} > 0$; 3) $f(x,t) \in L_2(Q_T)$. Under these conditions there exists a unique weak solution of problem (9), (10), periodic in t with period T.*

Let us note the main points of the proof, which uses Galerkin' s method. The n-th approximation is sought in the form of the n-th partial sum of series (3):

$$u_n(x,t) = X_0^{(n)}(x)d_0 + \sum_{k=1}^{n}\left(X_k^{(n)}(x)d_k(t) + Y_k^{(n)}(x)e_k(t)\right).$$

Substituting $u_n(x,t)$ into equation (9) and condition (10), and requiring that the results of these substitutions be orthogonal to each function of system (4) for $k \leq n$, we find that the system of functions $X_0^{(n)}(x)$, $X_k^{(n)}(x)$, and $Y_k^{(n)}(x)$ is determined as the solution of the class $\overset{0}{W}{}_2^2(D)$ of the first boundary-value problem for a strongly elliptic system of equations. The existence of such a solution follows from (5).

Next one establishes the uniform estimate $\|u_n\|_{W_2^1(Q_T)} \leq C$, ensuring weak compactness in $\overset{0}{W}{}_2^1(Q_T)$ of the sequence $\{u_n(x,t)\}$.

Let $\{u_{n_k}(x,t)\}$ be a weakly convergent subsequence of the sequence $\{u_n(x,t)\}$, and let $u(x,t)$ be its limit. Then, as is known, $u(x,t) \in \overset{0}{W}{}_2^1(Q_T)$. Since the sequence $\{u_{n_k}(x,t)\}$ converges uniformly with respect to $t \in [t_0, t_0+T]$ in $L_2(D)$ to $u(x,t)$, and

$$\|u(x,t_0+T) - u(x,t_0)\|_{L_2(D)} \leq \left\|u(x,t_0+T) - u_{n_k}(x,t_0+T)\right\|_{L_2(D)} + \left\|u_{n_k}(x,t_0) - u(x,t_0)\right\|_{L_2(D)},$$

then equality (11) is valid for $u(x,t)$.

If by $\Phi_n(x,t)$ we denote the n-th partial sum of the Fourier series of the function $\Phi(x,t) \in \overset{0}{W}{}_2^1(Q_T)$ with respect to system (4), then it is not difficult to verify the existence of the relations:

$$\int_{Q_T} \left(\frac{\partial u_{n_k}}{\partial t} \Phi_{n_k} + \sum_{i,j=1}^{m} a_{ij} \frac{\partial u_{n_k}}{\partial x_i} \frac{\partial \Phi_{n_k}}{\partial x_j} + c u_{n_k} \Phi_{n_k} - f \Phi_{n_k} \right) dt\, dx = 0,$$

$$\int_{Q_T} \left[\frac{\partial u_{n_k}}{\partial t} (\Phi - \Phi_{n_k}) + \sum_{i,j=1}^{m} a_{ij} \frac{\partial u_{n_k}}{\partial x_i} \frac{\partial (\Phi - \Phi_{n_k})}{\partial x_j} + \right.$$

$$\left. + c u_{n_k} (\Phi - \Phi_{n_k}) - f(\Phi - \Phi_{n_k}) \right] dt\, dx \underset{n_k \to \infty}{\longrightarrow} 0,$$

from which follows the validity of identity (12) for $u(x,t)$.

The uniqueness of the solution follows from (12), if one puts there $f(x,t) \equiv 0$, $u = u_1 - u_2$ and $\Phi = u_1 - u_2$.

The method of proof of Theorem 2, as is immediately clear, can be used in the investigation of weak periodic solutions of boundary-value problems for parabolic equations of higher orders.

Passing to periodic solutions of the Neumann problem, we formulate the result obtained.

Theorem 3. Let $\overline{D}$ belong to the class $A^{(2)}$, and let the coefficients of L be subject in D to the conditions: $a_{ij}(x) \in C^{(2,\lambda)}$, $b_i(x) \in C^{(1,\lambda)}$, $c(x) \in C^{(0,\lambda)}$,

$-c(x) \geq c_0$ ($c_0 = \text{const} > 0$). The functions $a(x,t)$, $\varphi(x,t)$ given on S are periodic in t with period T and have the following properties: these functions are continuous together with their derivatives with respect to t up to order 4 inclusive, and, moreover, $a(x,t) \geq a_0$, where $a_0 = \text{const} > 0$. Under the indicated conditions there exists a unique classical solution, periodic in t with period T, of the problem

$$u_t = Lu, \tag{13}$$

$$\left.(\partial u/\partial\gamma - a(x,t)u)\right|_S = \varphi(x,t), \tag{14}$$

where γ is the direction of the conormal to S.

Proof. Since the proof of this theorem, based on the use of Galerkin' s method, requires a considerable number of estimates, we shall restrict ourselves to the important case when $a(x,t) = a(x)$.

In the case under consideration the solution of the problem is sought in the form of the series (3), whose coefficients are determined from the sequence of boundary-value problems:

$$LX_0 = 0, \qquad \left.(\partial X_0/\partial\gamma - a(x)X_0)\right|_\Gamma = \varphi_0(x); \tag{15}$$

$$LX_k - c_{kY}k = 0, \qquad \left.(\partial X_k/\partial\gamma - a(x)X_k)\right|_\Gamma = \widetilde{\varphi}_k(x),$$

$$LY_k + c_{kX}k = 0, \qquad \left.(\partial Y_k/\partial\gamma - a(x)Y_k)\right|_\Gamma = \widetilde{\widetilde{\varphi}}_k(x) \quad (k = 1, 2, \ldots). \tag{16}$$

Leaving aside problem (15), which has a unique classical solution in D, let us turn to problems (16) and fix k. Since for the square of the length of the classical solution in D of problem (16) there are valid both the extremal property noted in paper [2], and a theorem analogous to Theorem I of paper [6], in $\overline{D}$ the estimate holds:

$$X_k^2 + Y_k^2 \leq \frac{1}{a_0} \max_{x\in\Gamma} \left(\widetilde{\varphi}_k^2 + \widetilde{\widetilde{\varphi}}_k^2\right). \tag{17}$$

The further proof of the theorem repeats the corresponding parts of the proof of Theorem 1. The uniqueness of the periodic solution of the Neumann problem follows from the periodicity of the solution, the maximum principle for equation (13), and Theorem 1 of paper [7].

Theorems 1 and 3 contain a justification, important for physical applications, of the method of finding periodic solutions of the Dirichlet and Neumann problems for second-order parabolic equations in the form of plane waves.

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Note: Figure translations are in progress. See original paper for figures.