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# MATHEMATICS

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**Abstract**

**Full Text**

## MATHEMATICS

**I. S. Kukles and A. M. Suyarshev**

### A GENERALIZED FROMMER METHOD

*(Presented by Academician I. G. Petrovskii, VII 8, 1960)*

In the present article we consider the differential equation

$$\frac{dy}{dx} = \frac{\alpha_0 y^m + \alpha_1(x) y^{m-1} + \alpha_2(x) y^{m-2} + \dots + \alpha_m(x)}{\beta_0 y^n + \beta_1(x) y^{n-1} + \beta_2(x) y^{n-2} + \dots + \beta_n(x)}, \quad (1)$$

where  $\alpha_0$  and  $\beta_0$  are constants ( $\alpha_0^2 + \beta_0^2 \neq 0$ ); the functions  $\alpha_i(x)$  and  $\beta_j(x)$  are differentiable for small  $x > 0$ , preserve their sign and vanish together with  $x$ , while in the case when all the functions  $\alpha_i(x)$  ( $i = 1, 2, \dots, m$ ) are identically equal to zero, at least one of the functions  $\beta_j(x)$  ( $j = 1, 2, \dots, n$ ) is not identically zero. We pose the problem: to determine whether there exist characteristics of equation (1) entering the origin from the right, and, if they exist, whether the set of these characteristics is finite or infinite. In addition, we pose the problem of estimating the order of smallness of solutions  $y(x)$  of equation (1) that vanish at the origin (if such solutions exist).

If the functions  $\alpha_i(x)$  and  $\beta_j(x)$  are analytic, then the stated problems are solved by means of Frommer's method (1, 2). In the present, more general case we shall use a more general method, from which Frommer's method follows as a special case.

Introducing into equation (1) the substitution  $y = u\omega(x)$ , where  $\omega(x)$  is a function differentiable for small  $x > 0$ , we shall have

$$\frac{du}{dx} = \frac{P(x, u)}{Q(x, u)}, \quad (2)$$

where

$$\begin{aligned} P(x, u) = & u^m \omega^{m-1} \left( \alpha_0 - \beta_0 \frac{\omega'}{\omega} \right) + u^{m-1} \omega^{m-2} \left( \alpha_1 - \alpha_1 \frac{\omega'}{\omega} \right) + \dots \\ & \dots + u \left( \alpha_m - \beta_{m-1} \frac{\omega'}{\omega} \right) + \frac{\alpha_m}{\omega}; \end{aligned} \quad (3)$$

$$Q(x, u) = \beta_0 u^{m-1} \omega^{m-1} + \beta_1(x) u^{m-2} \omega^{m-2} + \beta_2 u^{m-3} \omega^{m-3} + \dots + \beta_{m-1}(x). \quad (3')$$

Above we assumed  $m = n + 1$ , which does not restrict generality. Indeed, if  $m > n + 1$  ( $m = n + 1 + s$ ), then we shall assume that  $\alpha_0 = \alpha_1 = \dots = \alpha_{s-1} = 0$ , while if  $m < n + 1$  ( $m = n + 1 - s$ ), then we shall assume that  $\beta_0 = \beta_1 = \dots = \beta_{s-1} = 0$ .

Let us introduce the concepts of order and measure of smallness. Let  $y(x)$  be a continuous function defined for  $x > 0$ . If  $\lim_{x \rightarrow +0} \frac{y(x)}{\omega(x)} = 0$ , then we shall say that the function  $y(x)$  has an order of smallness higher than  $\omega(x)$ ; if  $\lim_{x \rightarrow +0} \left| \frac{y(x)}{\omega(x)} \right| = \infty$ , then  $y(x)$  has an order of smallness lower than  $\omega(x)$ . If  $\lim_{x \rightarrow +0} \frac{y(x)}{\omega(x)} = A$ , where  $0 < |A| < \infty$ , then  $y(x)$  has order of smallness  $\omega(x)$  and measure of smallness  $A$ .

If the quotient  $y(x)/\omega(x)$ , as  $x$  tends to zero from the right, remains uniformly bounded above and below, but does not tend to any limit, then  $y(x)$  has order of smallness  $\omega(x)$ , but has no definite measure of smallness. If the ratio  $y(x)/\omega(x)$  does not tend to any limit as  $x$  tends to zero from the right, and is not bounded above (or below), then the functions  $y(x)$  and  $\omega(x)$  are not comparable with one another with respect to order of smallness.

Considering only those of the functions  $\alpha(x)$  and  $\beta(x)$  which are not identically equal to zero, introduce the notation:

$$\omega_{ij} = \left| \frac{\alpha_j}{\alpha_i} \right|^{\frac{1}{j-i}}, \quad \omega_{i'j'} = \left| \frac{\beta_j}{\beta_i} \right|^{\frac{1}{j-i}},$$

$$\omega_{ij'} = \omega_{j'i} = \left[ (j-i) \int_{x_0}^x \left| \frac{\alpha_i}{\beta_i} \right| dx \right]^{\frac{1}{i-j}}, \quad \omega_{ii'} = \exp \left[ \int_{x_0}^x \left| \frac{\alpha_i}{\beta_i} \right| dx \right].$$

We shall assume henceforth that all the functions  $\omega$  are comparable with one another with respect to order of smallness.

We shall say that the term  $\gamma_j = \alpha_j \omega^{m-j-1}$  (or  $\gamma_{j'} = \beta_j \omega^{m-j-2} \omega'$ ) is situated to the right of  $\gamma_i = \alpha_i \omega^{m-i-1}$  (or, respectively,  $\gamma_{i'} = \beta_i \omega^{m-i-2} \omega'$ ) if  $j > i$ , and, in addition, the term  $\gamma_{i'}$  is situated to the right of  $\gamma_i$ . We shall also agree that the function  $\omega_{ik}$  is situated to the right of  $\omega_{ij}$  if  $k > j$  (an analogous property holds also for the functions  $\omega_{ik'}$ ,  $\omega_{i'k}$ , and  $\omega_{i'k'}$ ), and that the function  $\omega_{ik'}$  (or  $\omega_{i'k}$ ) is situated to the right of  $\omega_{ik}$  (or, respectively,  $\omega_{i'k}$ ).

Let us introduce the notion of characteristic functions. Let  $\gamma_i$  (or  $\gamma_{i'}$ ) be the leftmost of the terms  $\gamma$  not identically equal to zero. Considering the functions  $\omega_{ii'}$ ,  $\omega_{i,i+1}$ ,  $\omega_{i,(i+1)'}$ , ...,  $\omega_{im}$  (or, respectively,  $\omega_{i',i+1}$ ,  $\omega_{i',(i+1)'}$ ,  $\omega_{i',i+2}$ , ...,  $\omega_{i'm}$ ),

we shall call that one of them which has the least order of smallness (and, if there are several such functions, the rightmost of them) the first characteristic function.

If the first characteristic function is equal to  $\bar{\omega}_{ij}$  (or  $\bar{\omega}_{i'j'}$ ) (the bar over the letter  $\omega$  means that this function is characteristic), then we consider the functions  $\omega_{jj'}$ ,  $\omega_{j,j+1}$ ,  $\omega_{j,(j+1)'}$ , ...,  $\omega_{jm}$ , and that one of these functions which has the least order of smallness (and, if there are several such functions, the rightmost of them) we call the second characteristic function  $\bar{\omega}_{jk}$ , or  $\bar{\omega}_{jk'}$ , or  $\bar{\omega}_{jj'}$ . If the first characteristic function has the form  $\omega_{ij'}$ , then the second function will have the form  $\omega_{j'k}$  or  $\omega_{j'k'}$ . The third, fourth, etc. characteristic functions are defined analogously. The functions  $\omega_{kl}$ ,  $\omega_{kl'}$ , and  $\omega_{k'l'}$  will be called ordinary, and the function  $\omega_{kk'}$  special.

If, for  $\omega = \omega_{kl}$ , the terms  $\alpha_k \omega^{m-k-1}$  and  $\alpha_l \omega^{m-l-1}$  are equivalent for small  $|x|$ , then in place of the function  $\omega_{kl}$  we introduce the function  $\bar{\omega}_{kl}$ , defined as follows:

$$\omega_{kl} = \left( \exp \int_{x_0}^x \frac{\alpha_i}{\beta_i} dx \right) \left| \int_{x_0}^x \alpha_j \exp \left[ (i-j) \int_{x_0}^x \frac{\alpha_i}{\beta_i} dx \right] dx \right|^{\frac{1}{j-i}}.$$

We call the function  $\bar{\omega}_{kl}$  a right or left quasi-special function according as  $k < l$  or  $k > l$ .

If, for  $\omega = \omega_{kl'}$ , the terms  $\alpha_k \omega^{m-k-1}$  and  $\beta_l \omega^{m-l-2} \omega'$  are equivalent, then in place of the function  $\omega_{kl'}$  we introduce the function  $\bar{\omega}_{kl'}$ , defined as follows:

$$\omega_{kl'} = \left( \exp \int_{x_0}^x \frac{\alpha_i}{\beta_i} dx \right) \left| \int_{x_0}^x \frac{\alpha_i \beta_j}{\beta_i^2} \exp \left[ (i-j) \int_{x_0}^x \frac{\alpha_i}{\beta_i} dx \right] dx \right|^{\frac{1}{j-i}}.$$

We also call this function a right or left quasi-special function according as  $k < l$  or  $k > l$ . It may, of course, happen that one or several quasi-special functions are characteristic.

The total number of characteristic functions does not exceed the number  $m$ . It has been proved that every  $i$ -th characteristic function has an order of smallness higher than that of the preceding  $(i-1)$ -st characteristic function.

**Theorem 1.** *Every solution  $y(x)$  of equation (1), defined in the right half-plane and vanishing at the origin, has an order of smallness coinciding with the order of one of the characteristic functions.*

If the characteristic function  $\bar{\omega}$  is ordinary, then equation (2) has the form

$$\frac{du}{dx} = \frac{N(u) + \varepsilon(x, u)}{k(u) [N_1(u) + \varepsilon_1(x, u)]}, \quad (4)$$

where  $N(u)$  and  $N_1(u)$  are polynomials, with  $N(u)$  containing at least two terms and  $N_1(u)$  at least one term;  $\varepsilon(x, u)$ ,  $\varepsilon_1(x, u)$ , and  $k(x)$  are continuous functions vanishing together with  $x$ , and  $k(x)$  satisfies the condition

$$\int_0^x \frac{dx}{k(x)} = \infty.$$

**Theorem 2.** *If the order of smallness of a solution  $y(x)$  of equation (1) coincides with the order of an ordinary characteristic function  $\bar{\omega}$ , then the measure of smallness of this solution is equal to one of the real nonzero roots of the equation  $N(u) = 0$  (the equation of measures of smallness corresponding to the given order of smallness  $\bar{\omega}$ ).*

As a consequence of Theorem 2 it follows that if the equation of measures of smallness  $N(u) = 0$  has no real nonzero roots, then in the right half-plane there exist no solutions  $y(x)$  of equation (1) having the given order of smallness  $\bar{\omega}$ .

If  $\bar{\omega}$  is a special characteristic function, then equation (2) will have the form

$$\frac{du}{dx} = \frac{N_2(u) + \varepsilon(x, u)}{\lambda(u) [N_1(u) + \varepsilon_1(x, u)]}, \quad (4')$$

where  $N_2(u)$  and  $N_1(u)$  are polynomials not identically equal to zero;  $\varepsilon(x, u)$  and  $\varepsilon_1(x, u)$  are continuous functions vanishing together with  $x$ ; the function  $\lambda(x)$  is continuous for small  $x \geq 0$ , but may have a discontinuity at the point  $x = 0$ .

**Theorem 3.** *If  $\bar{\omega}_{hk'}$  is a special characteristic function, then three cases are possible: 1)  $\alpha_k(x)$  and  $\beta_k(x)$  have opposite signs; 2)  $\alpha_k(x)$  and  $\beta_k(x)$  have the same signs, and the integral*

$$\int_0^x \frac{dx}{\lambda(x)} \quad (5)$$

*(where  $\lambda(x)$  is the function appearing in the denominator of the right-hand side of equation (4')) diverges; 3)  $\alpha_k(x)$  and  $\beta_k(x)$  have the same signs and the integral (5) converges.*

In the first case there are no solutions of equation (1) having order of smallness  $\bar{\omega}_{hk'}$  in the right half-plane. In the second case such solutions exist only when their measures of smallness are equal to real nonzero roots of the equation of measures of smallness  $N_2(u) = 0$ . In the third case there exists an infinite set of solutions having order of smallness  $\bar{\omega}_{hk'}$ , and each of these solutions has its own measure of smallness (a special case).

Let  $\delta$  be one of the real nonzero roots of the equation of measures of smallness  $N(u) = 0$ , and let  $r$  be the multiplicity of this root. Setting  $u - \delta = \bar{u}$  in equation (4), we bring it to the form

$$\frac{d\bar{u}}{dx} = \frac{\bar{u}^r \bar{N}_2(\bar{u}) + \varepsilon(x, \bar{u} + \delta)}{k(x) [N_1(\bar{u} + \delta) + \varepsilon_1(x, \bar{u} + \delta)]}, \quad (6)$$

where  $\bar{N}(\bar{u})$  is a polynomial not vanishing for  $\bar{u} = 0$ . Let  $N_1(\delta) \neq 0$ . Then equation (6) is a generalized (for  $r > 1$ ) or ordinary (for  $r = 1$ ) Briot–Bouquet equation. If odd  $r \geq 1$  and  $\rho = \bar{N}(0)/N_1(\delta) = 0$ , then there exists an infinite set of solutions of equation (1) having order of smallness  $\bar{\omega}$  and measure of smallness  $\delta$ . If  $r = 1$  and  $\rho < 0$ , then there exists (in the right half-plane) only one such solution. The question remains open for odd  $r > 1$  and  $\rho < 0$ , and also for even  $r$ . If in these cases the problem is not solved by means of one of the known criteria (see (3)), then we introduce a new substitution  $\bar{u} = u_1 \omega_1(x)$  and begin the process considered above anew. In doing so it may happen that we again arrive at a generalized Briot–Bouquet equation and must introduce a new substitution. The question arises of the number of such substitutions necessary for reducing equation (1) to an ordinary Briot–Bouquet equation, and of the very possibility of such a reduction. This question, solved for the case of analytic functions  $\alpha_i(x)$  and  $\beta_i(x)$  <sup>(4)</sup>, remains open in the present case.

Uzbek State University  
named after Alisher Navoi

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*Note: Figure translations are in progress. See original paper for figures.*

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