On Some Problems for Equations of the Theory of Combustion

1961

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Abstract

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MATHEMATICS

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On Some Problems for Equations of the Theory of Combustion

(Presented by Academician I. G. Petrovskii, 25 VII 1960)

In the theory of combustion ([1]), the Cauchy problem is considered for the equation

$$\partial u/\partial t - \partial^2 u/\partial x^2 = F(u) \tag{1}$$

with the initial condition

$$u\big|_{t=0} = u_0(x). \tag{2}$$

In the present work we investigate the behavior of the solution of problem (1)–(2) as $t \to \infty$, under certain natural assumptions concerning $F(u)$ and $u_0(x)$.

Theorem 1 is a generalization of the main theorem of ([2]). Theorems 2–4 generalize and refine the results of ([3]). In Theorems 5–7 the case of a finite, i.e. equal to zero outside a finite interval, function $u_0(x)$ is considered. Related questions are also the subject of works ([4–7]). Below, throughout, sufficient smoothness of $F(u)$ is assumed.

Theorem 1. Let $u(x,t)$ be the solution of problem (1)–(2), where

$$F(0) = F(1) = 0; \quad F(u) \leqslant 0 \text{ for } 0 < u < a < 1,$$

$$F'(u) \leqslant 0 \text{ for } 0 < u < a_0 \leqslant a; \quad F(u) > 0 \text{ for } a < u < 1; \tag{3}$$

$$F'(1) < 0; \quad \int_0^1 F(u)\,du > 0;$$

the function $u_0(x)$ does not decrease and, for some $x_1 \leqslant x_2$,

$$u_0(x) = 0 \text{ for } x < x_1; \qquad u_0(x) = 1 \text{ for } x > x_2. \tag{4}$$

Let $\tilde{u}(x + mt + C)$ be a stationary solution of equation (1). Then, for some $C = C^0$,

$$\left|u(x,t) - \tilde{u}(x + mt + C^0)\right| \to 0$$

as $t \to \infty$, uniformly in x varying over the entire number axis.

Proof. We pass to the coordinates $x' = x + mt$, $t' = t$, denoting them again by x and t. Equation (1) takes the form

$$\partial u/\partial t - \partial^2 u/\partial x^2 + m\partial u/\partial x = F(u). \tag{1'}$$

In the half-plane $t > 0$, represent the solution $u(x,t)$ of problem $(1')$–(2) in the form $u(x,t) = \tilde{u}(x + C(x,t))$. Since $\tilde{u}'(x + C) > 0$, $\tilde{u}(-\infty) = 0$, $\tilde{u}(+\infty) = 1$ ([1,2]), and for $t > 0$, $0 < u(x,t) < 1$ ([7]), such a representation is possible. For $C(x,t)$ we obtain the equation

$$\partial C/\partial t - \partial^2 C/\partial x^2 + b(x,t)\,\partial C/\partial x = 0; \tag{5}$$

$$b(x,t) = m - p'(u(x,t))\left[1 + \frac{\partial u}{\partial x}/p(u)\right], \tag{6}$$

where, by definition,

$$p(u(x,t)) = p[\tilde{u}(x + C(x,t))] = \tilde{u}'(x + C(x,t)).$$

It can be shown that

$$p(0) = p(1) = 0; \quad p'(0) > 0; \quad p'(1) < 0; \quad p(u) > 0 \quad \text{for } 0 < u < 1. \tag{7}$$

Now it remains to prove that as $t \to \infty$ there exists $\lim C(x,t) = \text{const} \neq \infty$.

Let $\bar{u}(x,t)$ be the solution of problem $(1')$–(2), where $u_0(x) = 0$ for $x < 0$; $u_0(x) = 1$ for $x > 0$. From the maximum principle it follows that

$$\bar{u}(x - x_2, t) \le u(x,t) \le \bar{u}(x - x_1, t). \tag{8}$$

Without loss of generality one may additionally assume that

$$0 < u_0(x_1 + 0) \le u_0(x_2 - 0) < 1. \tag{9}$$

Taking into account (8), the convergence, uniform in all x, of $\bar{u}(x,t)$ to a certain stationary solution (2), and (9), by means of Bernstein-type estimates (8, 9) and barrier functions one can prove that in the half-plane $t > t_0 > 0$

$$C(x,t) = C_1(x,t) - C_2(x,t). \tag{10}$$

Here C_1, C_2 satisfy equation (5), do not decrease with respect to x, have, together with their derivatives, power-order growth in x; C_2 is bounded in the half-plane $t > t_0$; C_1 is bounded in any half-strip $|x| < N,\ t > t_0$, and one can indicate such an x_0 that for $x > x_0,\ t > t_0$

$$|C_1| < e^{\varepsilon x} + K, \quad 0 < \varepsilon < m, \quad K = \text{const}. \tag{11}$$

Since $u_0(x)$ does not decrease, $\partial u/\partial x \geq 0$. Then, also taking into account (6), (7), (8), we may assert that

$$b(x,t) > m \quad \text{for } x > x_0,\ t > t_0 \tag{12}$$

and that $b(x,t)$ is bounded in any half-strip $|x| < N,\ t > t_0$. Therefore one can construct a continuous function $\bar{b}(x)$ such that $\bar{b}(x) < b(x,t)$ in the half-plane $t > t_0$, $\bar{b}(x) > m$ for $x > x_0$.

Multiplying equation (5) for C_i by

$$f(x) = \exp\left(-\int_0^x \bar{b}\, d\xi\right)$$

and integrating with respect to x from x to $+\infty$, while taking into account the properties of $\bar{b}$ and C_i, we obtain for $t > t_0$

$$\frac{d}{dt}\int_x^{+\infty} C_i(\xi,t) f(\xi)\, d\xi = -f(x)\frac{\partial C_i}{\partial x} - \int_x^{+\infty} f\frac{\partial C_i}{\partial \xi}(b - \bar{b})\, d\xi \leq 0, \quad i = 1, 2. \tag{13}$$

Inequalities (13) and (11) ensure the weak convergence of $C(x,t) = C_1 - C_2$ as $t \to \infty$. Taking also into account Bernstein-type estimates, we then obtain that $C(x,t)$, as $t \to \infty$, converges uniformly on every finite interval of variation of x to a certain limiting function $C^0(x)$, satisfying the equation $-C'' + \bar{b}(x)C' = 0$, where $\bar{b}(x) > m$ for $x > x_0$. Since, by virtue of (8), $C^0(x)$ is moreover bounded, it follows that $C^0(x) = \text{const}$.

We shall call $u_0(x)$ a perturbed stationary solution if $u_0(x) = \tilde{u}(x + C_0(x))$, where $C_0(x)$ is a function of bounded variation on the entire number axis.

Theorem 2. Let $F(u)$ satisfy conditions (3) or the conditions (cf. (7))

$$F(0) = F(1) = 0; \quad F'(0) > 0; \quad F'(1) < 0; \tag{14}$$

$$F'(u) \le F'(0), \quad F(u) > 0 \quad \text{for } 0 < u < 1,$$

$u_0(x) = \tilde{u}(x + C_0(x))$ is a perturbed stationary solution, additionally satisfying one of the conditions: a) $u_0(x)$ is nondecreasing; b) $|[u_0(x) - \tilde{u}(x)]/\tilde{u}'(x)| < \varepsilon$, where ε is sufficiently small.

Then the assertion of Theorem 1 is valid.

The proof is analogous to that given above. Here, in the expansion (10) for $t = 0$, C_1, C_2 are nondecreasing and bounded. Instead of (8) the inequalities used are

$$\tilde{u}(x + C') < u(x,t) < \tilde{u}(x + C''), \qquad C', C'' = \text{const.} \tag{8'}$$

Condition a) or b) ensures the validity of inequality (12) for $t \ge t_0 > 0$. Thus, in the presence of condition b), with the aid of estimates of Bernstein type one can show that

$$b(x,t) = m - p'(u)(2 + O(\varepsilon)) \qquad \text{for } t \ge t_0 > 0. \tag{15}$$

From (15), taking into account (8) and (7), for sufficiently small ε we obtain inequality (12).

It is of interest to note that if $F(u)$ satisfies conditions (3), and $F(u) = 0$ for $0 < u < \alpha$, then one can construct a function $u_0(x)$, monotone, equal to unity for $x > 0$, tending to zero as $x \to -\infty$, such that the solution of problem (1)–(2) does not tend, as $t \to \infty$, to any stationary solution.

Theorem 3. Let $F(u)$ satisfy conditions (3); let $u_0(x) = \tilde{u}(x) + v_0(x)$ satisfy the conditions of Theorem 2; $v_0(x) \ge 0$ (≤ 0); and $v_0(x) > 0$ (< 0) on some interval.

Then the solution of problem (1)–(2) tends as $t \to \infty$ to the "shifted" stationary solution $\tilde{u}(x + C^0)$, where $C^0 > 0$ (< 0).

The theorem follows from the strong maximum principle (10) and the following considerations. If $F(u)$ satisfies conditions (3), then

$$p'(0) = m/2 + \sqrt{m^2/4 - F'(0)}, \qquad p'(1) = m/2 - \sqrt{m^2/4 - F'(1)}.$$

From these equalities, as well as from (15) and (8′), it follows that, under condition b) of Theorem 2, for some $\gamma > 0$, $x_0 > 0$, and sufficiently small ε in (15), $b(x,t) < -\gamma$ for $x < -x_0$, $t \ge t_0 > 0$; and $b(x,t) > \gamma$ for $x > x_0$, $t \ge t_0$.

Theorem 4. Let $F(u)$ satisfy conditions (14); let $u(x,t)$, $v(x,t)$ be solutions of the Cauchy problem for equation (1) with initial functions $u_0(x)$, $v_0(x)$, respectively. Let $0 \le u_0, v_0 \le 1$, $u_0 - v_0 = O[\exp(\frac{1}{2}mx - \varepsilon|x|^{\alpha})]$ as $x \to -\infty$, where $\varepsilon > 0$, $1/2 < \alpha \le 1$, and, for some $C', C'' = \text{const}$,

$$\tilde{u}(x+C') < u_0 < \tilde{u}(x+C''), \qquad \tilde{u}(x+C') < v_0 < \tilde{u}(x+C'').$$

Then $|u(x,t) - v(x,t)| \to 0$ as $t \to \infty$, uniformly in x varying over the entire number axis.

The theorem is proved with the aid of the maximum principle and the Poisson formula for the solution of the Cauchy problem for the heat equation.

In Theorems 5–7, the convergence of the solution $u(x,t)$ of problem (1)–(2) as $t \to \infty$ to its limit is understood in the sense of convergence uniform with respect to x varying over any finite interval.

Lemma. Let

$$F(0) = F(1) = 0; \qquad F(u) \ge 0 \text{ for } 0 \le u \le 1;$$

$$u_0(x) = 0 \text{ for } |x| > l; \qquad u_0(x) = 1 \text{ for } |x| < l. \tag{16}$$

Then, as $t \to \infty$, $u(x,t) \to c_0$, where $0 \le c_0 \le 1$, $F(c_0) = 0$.

In the proof of the lemma one uses the fact that the set of points in the x,t-plane at which $\Delta u = u(x,t+\Delta t) - u(x,t) < 0$ for $\Delta t > 0$ is a simply connected domain adjoining the interval $|x| < l$ of the x-axis (cf. (7), Theorem 11).

Theorem 5. Let $F(u)$ satisfy conditions (3); $F(u) = 0$ for $0 < u < \alpha$; the numbers $\tilde{\alpha}, y_0$ are such that $\alpha < \tilde{\alpha} < y_0 < 1$, and for $\tilde{\alpha} < u < y_0$

$$F(u) > k(u - \tilde{\alpha}), \quad \text{where} \quad k = F(y_0)/(y_0 - \tilde{\alpha}).$$

Let $u_0(x)$ be of the form (16), where

$$l > \pi/2\sqrt{k} + \tilde{\alpha}/\sqrt{k}(y_0 - \tilde{\alpha}).$$

Then $u(x,t) \to 1$ as $t \to \infty$.

The theorem is proved with the aid of a lemma and by estimating the solution of problem (1)–(2) from below by a certain self-similar solution of the equation

$$\partial u/\partial t - \partial^2 u/\partial x^2 = \lambda^2 F_1(u)/(t + \lambda^2).$$

Here $F_1(u) = 0$ for $0 < u < \tilde{\alpha}$; $F_1(u) = k(u - \tilde{\alpha})$ for $\tilde{\alpha} < u < y_0$.

Theorem 6. Let $F(u) > 0$ for $0 < u < 1$; $F(1) = 0$; $F'(0) + F(0) > 0$; and let $u_0(x) > 0$ on some interval.

Then $u(x,t) \to 0$ as $t \to \infty$.

The theorem is proved with the aid of the maximum principle and a lemma.

Theorem 7. Let $F(u)$ satisfy conditions (3), and let $u_0(x)$ be of the form (16), where

$$l < \sqrt{\pi/2q}, \quad q = \sup[F/u], \quad 0 \le u \le 1.$$

Then $u(x,t) \to 0$ as $t \to \infty$.

Indeed, from the maximum principle it follows that

$$0 \le u(x,t) \le \overline{u}(x,t) \exp qt \le \overline{u}(0,t) \exp qt.$$

Here $\overline{u}$ is the solution of the Cauchy problem for the heat equation, which becomes $u = u_0$ at $t = 0$. The required estimate for l is obtained from the requirement that, for some $t = t_0 > 0$,

$$\overline{u}(0,t) \exp qt < \alpha.$$

Since $u = \alpha$ is a solution of equation (1), by the maximum principle $u(x,t) < \alpha$ for $t > t_0$, and therefore $F(u) \le 0$ for $t > t_0$. From this it is not difficult to obtain the assertion of the theorem.

Let us note additionally that if $F(u)$ satisfies conditions (3), then it can be majorized by a function $F_1(u)$, where $F_1(u) = 0$ for $0 < u < \alpha_1 < \alpha$; $F_1(u) = k(u-\alpha_1)^{1+n}$, $n \ge 1$, $k > 0$, for $\alpha_1 \le u \le 1$.

The equation

$$\partial u/\partial t - \partial^2 u/\partial x^2 = F_1(u)$$

has self-similar solutions of the form

$$\alpha_1 + \sqrt[n]{\lambda^2}\, y\left(x/\sqrt{\lambda^2+t}\right) / \sqrt[n]{\lambda^2+t}.$$

Then, for any finite $u_0(x) \le \alpha_1 + y(x/\lambda)$, the solution of problem (1)–(2) tends to zero as $t \to \infty$.

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Received
23 VII 1960

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Note: Figure translations are in progress. See original paper for figures.