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Abstract

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I. I. Bavrin

ON THE COEFFICIENTS OF A CLASS OF ANALYTIC FUNCTIONS OF TWO COMPLEX VARIABLES

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Let $D \ni (0,0)$ be a bounded complete bicircular domain whose boundary is twice continuously differentiable and analytically convex. As A. A. Temlyakov proved $^{(1)}$, the boundary of this domain is parametrically given in the form $|w| = r_1(\tau)$, $|z| = r_2(\tau)$, $0 \le \tau \le 1$, where $r_1(0) = 0$, $r_1(1) < \infty$, $r_1'(\tau) > 0$ in $(0,1]$,

$$r_2(\tau) = \exp\left[-\int_\infty^\tau \frac{\tau}{1-\tau}\, d\ln r_1(\tau)\right] \quad (r_2(1) = 0^{(1)}).$$

C_E (C_D) is the class of functions

$$F(w,z) = \sum_{m,n=0}^{\infty} a_{mn} w^m z^n \quad (a_{00} = 0),$$

regular in the bicylinder $E\{|w| < R_1,\ |z| < R_2\}$ (in the domain D) and satisfying, for any (w_1, z_1) and (w_2, z_2) from E (D), the condition $F(w_1, z_1)F(w_2, z_2) \neq 1$.

Let the function

$$F(w,z) = \sum_{m,n=0}^{\infty} a_{mn} w^m z^n$$

satisfy the following conditions: a) it is regular in E; b) for $0 < \rho < 1$ the integral

$$\int_0^{2\pi} |\psi(\rho e^{i\varphi}, t_0)|\, d\varphi$$

is bounded, where

$$\psi(\rho e^{i\varphi}, t_0) = F(R_1, \rho e^{i\varphi}, R_2 \rho e^{i(\varphi - t_0)})$$

and t_0 is any fixed value from $[0, 2\pi]$. We shall denote this class of functions by q. According to condition a), the function $F(\rho w, \rho z)$ $(0 < \rho < 1)$ is regular in the closed bicylinder $\overline{E}$, and, consequently, by virtue of the integral formula of A. A. Temlyakov (2),

$$a_{mn}\rho^{m+n} = (4\pi^2 R_1^m R_2^n)^{-1} \int_0^{2\pi} dt \int_0^{2\pi} \psi(\rho e^{i\varphi}, t)\, e^{-i[(m+n)\varphi - nt]}\, d\varphi. \qquad (1)$$

From condition b) it follows that the modulus of the integral

$$\int_0^{2\pi} \psi(\rho e^{i\varphi}, t) e^{-i[(m+n)\varphi - nt]}\, d\varphi,$$

where $0 \le t \le 2\pi$, is bounded for $0 < \rho < 1$. From conditions a) and b) it follows that the function $\psi(\zeta, t_0) \in H_1$ and therefore has, almost everywhere on the circle $|\zeta| = 1$, definite boundary values along nontangential paths, forming the boundary function $\psi(e^{i\varphi}, t_0)$, and

$$\psi(\zeta, t_0) = (2\pi i)^{-1} \int_0^{2\pi} \psi(e^{i\varphi}, t_0)(e^{i\varphi} - \zeta)^{-1}\, de^{i\varphi},$$

where the integral is understood in the Lebesgue sense. Hence

$$[(m+n)!]^{-1} \psi^{(m+n)}(0, t_0) = (2\pi)^{-1} \int_0^{2\pi} \psi(e^{i\varphi}, t_0) e^{-i(m+n)\varphi}\, d\varphi. \qquad (2)$$

On the other hand, the function $\psi(\rho\xi, t_0)$ is regular in $|\xi| \leqslant 1$, and therefore, by Cauchy' s formula,

$$[(m+n)!]^{-1} \rho^{m+n} \psi^{(m+n)}(0, t_0) = (2\pi)^{-1} \int_0^{2\pi} \psi(\rho e^{i\varphi}, t_0) e^{-i(m+n)\varphi}\, d\varphi. \qquad (3)$$

From the equalities (2), (3) we find that

$$\lim_{\rho \to 1} \int_0^{2\pi} \psi(\rho e^{i\varphi}, t_0) e^{-i(m+n)\varphi}\, d\varphi = \int_0^{2\pi} \psi(e^{i\varphi}, t_0) e^{-i(m+n)\varphi}\, d\varphi,$$

and since t_0 is any fixed point of $[0, 2\pi]$, we have

$$\lim_{\rho\to 1}\int_0^{2\pi}\psi(\rho e^{i\varphi},t)e^{-i(m+n)\varphi}\,d\varphi=\int_0^{2\pi}\psi(e^{i\varphi},t)e^{-i(m+n)\varphi}\,d\varphi.$$

Passing in (1) to the limit as $\rho\to 1$, we obtain the formulas

$$a_{mn}=(4\pi^2R_1^mR_2^n)^{-1}\int_0^{2\pi}dt\int_0^{2\pi}\psi(e^{i\varphi},t)e^{-i[(m+n)\varphi-nt]}\,d\varphi,\tag{4}$$

where the integrals are understood in the sense of Lebesgue. Thus the following has been proved:

Theorem 1. For functions $F(w,z)=\sum\limits_{m,n=0}^{\infty}a_{mn}w^mz^n\in q$, the formulas (4) hold.

We shall say that a certain property L holds for almost all points (t,φ) of the square $S\{0\leqslant t\leqslant 2\pi,\ 0\leqslant\varphi\leqslant 2\pi\}$, or, briefly, almost everywhere on S, if almost all t with $0\leqslant t\leqslant 2\pi$ have the property that for them almost all φ with $0\leqslant\varphi\leqslant 2\pi$ possess the property L.

Corollary 1. If a function $F(w,z)\in q$ is such that $\psi(e^{i\varphi},t)=0$ almost everywhere on S, then $F(w,z)=0$ in E.

Lemma. If a function $F(w,z)\in C_E$, then

$$(4\pi^2)^{-1}\int_0^{2\pi}dt\int_0^{2\pi}|\psi(e^{i\varphi},t)|\,d\varphi\leqslant 1,\tag{5}$$

where the equality sign occurs only for functions $F(w,z)\in C_E$ for which $|\psi(e^{i\varphi},t)|=1$ almost everywhere on S.

Proof. Since $F(w,z)\in C_E$, the function $\psi(\xi,t)$, for every fixed t in $[0,2\pi]$, is regular in the disk $|\xi|<1$ and satisfies, for any ξ_1 and ξ_2 from this disk, the condition $\psi(\xi_1,t)\psi(\xi_2,t)\neq 1$. Therefore, by the corresponding proposition for one variable (3),

$$(2\pi)^{-1}\int_0^{2\pi}|\psi(e^{i\varphi},t)|\,d\varphi\leqslant 1,\tag{6}$$

whence the estimate (5) follows. In view of (6), equality in (5), as is known, is obtained only when

$$(2\pi)^{-1}\int_0^{2\pi}|\psi(e^{i\varphi},t)|\,d\varphi=1$$

for almost all t in $[0, 2\pi]$. This last equality, for each of the indicated almost all t, according to the proposition for one variable just mentioned, holds only when $|\psi(e^{i\varphi}, t)| = 1$ for almost all φ in $[0, 2\pi]$. Thus the equality sign in (5) occurs only for functions $F(w, z) \in C_E$ for which $|\psi(e^{i\varphi}, t)| = 1$ almost everywhere on S.

Remark 1. It follows from (6) that $C_E \subset q$.

Theorem 2. If the function $F(w, z) = \sum_{m,n=0}^{\infty} a_{mn} w^m z^n$ $(a_{00} = 0)$ belongs to C_E, then for $m + n > 0$

$$|a_{mn}| \leq R_1^{-m} R_2^{-n}\,{}^*; \tag{7}$$

equality occurs only for the function

$$F(w, z) = \varepsilon_{mn} \frac{w^m z^n}{R_1^m R_2^n}, \qquad |\varepsilon_{mn}| = 1. \tag{8}$$

Proof. According to formulas (1) and the lemma we have

$$\begin{aligned} |a_{mn}| = (4\pi^2 R_1^m R_2^n)^{-1} \left| \int_0^{2\pi} dt \int_0^{2\pi} \psi(e^{i\varphi}, t) e^{-i[(m+n)\varphi - nt]} \, d\varphi \right| \leq \\ \leq (4\pi^2 R_1^m R_2^n)^{-1} \int_0^{2\pi} \left| \int_0^{2\pi} \psi(e^{i\varphi}, t) e^{-i[(m+n)\varphi - nt]} \, d\varphi \right| dt \leq \\ \leq (4\pi^2 R_1^m R_2^n)^{-1} \int_0^{2\pi} dt \int_0^{2\pi} |\psi(e^{i\varphi}, t)| \, d\varphi \leq R_1^{-m} R_2^{-n}. \end{aligned} \tag{9}$$

Using the proposition for one complex variable ** and the lemma, we conclude that the equality signs in (9) occur only in the case when

$$\arg \psi(e^{i\varphi}, t) e^{-i[(m+n)\varphi - nt]} \equiv \text{const} \pmod{2\pi}$$

and $|\psi(e^{i\varphi}, t)| = 1$ almost everywhere on S, which is equivalent to the single condition

$$\frac{R_1^m R_2^n F(R_1 e^{i\varphi}, R_2 e^{i(\varphi - t)})}{(R_1 e^{i\varphi})^m (R_2 e^{i(\varphi - t)})^n} = \text{const} = e^{i\alpha} = \varepsilon_{mn}$$

almost everywhere on S, which, by virtue of Corollary 1, is fulfilled only in the case of the function (8), for which equality occurs in (7).

Theorem 3. If the function $F(w,z)=\sum\limits_{m,n=0}^{\infty} a_{mn}w^m z^n$ $(a_{00}=0)$ belongs to C_D, then for $m+n>0$ we have the sharp estimates

$$|a_{mn}| \leq r_1^{-m}\left(\frac{m}{m+n}\right) r_2^{-n}\left(\frac{m}{m+n}\right) \tag{10}$$

(taking $0^0=1$); equality for $m>0,\ n>0$ occurs only for the function

$$F(w,z)=\varepsilon_{mn}\frac{w^m z^n}{r_1^m\left(\dfrac{m}{m+n}\right) r_2^n\left(\dfrac{m}{m+n}\right)}, \qquad |\varepsilon_{mn}|=1. \tag{11}$$

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* Hence, in particular, for one variable we obtain the known estimate (3).

** The equality sign in

$$\left|\int_0^{2\pi} f(e^{i\varphi})\,d\varphi\right| \leq \int_0^{2\pi} |f(e^{i\varphi})|\,d\varphi,$$

where $f(e^{i\varphi})$ and $|f(e^{i\varphi})|$ are summable on $[0,2\pi]$, is attained if and only if

$$\arg f(e^{i\varphi}) \equiv \text{const} \pmod{2\pi}$$

for almost all $\varphi,\ 0\leq\varphi\leq 2\pi$. Indeed, let

$$\theta=\arg\int_0^{2\pi} f(e^{i\varphi})\,d\varphi.$$

Then

$$\left|\int_0^{2\pi} f(e^{i\varphi})\,d\varphi\right| = e^{-i\theta}\int_0^{2\pi} f(e^{i\varphi})\,d\varphi = \int_0^{2\pi} |f(e^{i\varphi})|\cos[\arg f(e^{i\varphi})-\theta]\,d\varphi \leq \int_0^{2\pi} |f(e^{i\varphi})|\,d\varphi,$$

from which the assertion easily follows.

Proof. Since the hypersurface $|w|=r_1(\tau),\ |z|=r_2(\tau)$ is composed of the surfaces $|w|=r_1(\tau),\ |z|=r_2(\tau)$ under continuous variation of the parameter τ in the segment $[0,1]$, and since the point $(0,0)$ is an interior point of the bicylinder $|w|<r_1(\tau),\ |z|<r_2(\tau)$ for every $\tau,\ 0<\tau<1$, it follows, on the basis of Theorem 2, that

$$|a_{mn}| \le r_1^{-m}(\tau) r_2^{-n}(\tau), \qquad m > 0, \quad n > 0,$$

where τ is an arbitrary number from $(0, 1)$. Using functions of one complex variable

$$F(0, z) = \sum_{n=0}^{\infty} a_{0n} z^n, \qquad F(w, 0) = \sum_{m=0}^{\infty} a_{m0} w^m,$$

we have

$$|a_{0n}| \le r_2^{-n}(0), \qquad n > 0; \qquad |a_{m0}| \le r_1^{-m}(1), \qquad m > 0.$$

It is not difficult to see that

$$\max_{0 \le \tau \le 1} r_1^m(\tau) r_2^n(\tau) = \begin{cases} r_1^m\left(\dfrac{m}{m+n}\right) r_2^n\left(\dfrac{m}{m+n}\right), & m > 0, \quad n > 0; \\ r_1^m(1), & m > 0, \quad n = 0; \\ r_2^n(0), & m = 0, \quad n > 0; \end{cases}$$

therefore we obtain the estimate (10). Further, in the case of the bicylinder

$$\left\{ |w| < r_1\left(\frac{m_0}{m_0 + n_0}\right), \ |z| < r_2\left(\frac{m_0}{m_0 + n_0}\right) \right\}$$

$(m_0, n_0$ are any fixed values among $m = 1, 2, \dots; \ n = 1, 2, \dots)$, where the function $F(w, z)$ satisfies all the conditions of Theorem 2, the equality sign in the estimate

$$|a_{m_0 n_0}| \le r_1^{-m_0}\left(\frac{m_0}{m_0 + n_0}\right) r_2^{-n_0}\left(\frac{m_0}{m_0 + n_0}\right),$$

by virtue of Theorem 2, occurs only for the function

$$F(w, z) = \varepsilon_{m_0 n_0} \frac{w^{m_0} z^{n_0}}{r_1^{m_0}\left(\dfrac{m_0}{m_0 + n_0}\right) r_2^{n_0}\left(\dfrac{m_0}{m_0 + n_0}\right)}, \qquad |\varepsilon_{m_0 n_0}| = 1,$$

and since m_0, n_0 are any fixed values among $m = 1, 2, \dots; \ n = 1, 2, \dots$, equality in the estimate (10) for $m > 0, \ n > 0$ occurs only for the function (11).

Remark 2. In the case of n complex variables all the arguments are carried out in an entirely analogous way (for a polycylinder—as for a bicylinder, and for special classes of domains (4,5)—as for the domain D).

Moscow Regional Pedagogical Institute
named after N. K. Krupskaya

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Note: Figure translations are in progress. See original paper for figures.