MATHEMATICS

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

I. A. IBRAGIMOV

ON THE SPECTRAL FUNCTIONS OF SOME CLASSES OF STATIONARY GAUSSIAN PROCESSES

(Presented by Academician A. N. Kolmogorov on 25 XI 1960)

1. Let $x(t)$ be a stationary Gaussian process. Denote by $\mathfrak{M}_a^b$ the σ-algebra of events generated by the random variables $x(t)$, $a \leq t \leq b$. It is said that the process $x(t)$ has the property of **strong mixing** ([1]) if, as $\tau \to \infty$,

$$\sup_{\substack{A \in \mathfrak{M}_{-\infty}^0 \\ B \in \mathfrak{M}_\tau^\infty}} |\mathbf{P}(AB) - \mathbf{P}(A)\mathbf{P}(B)| = \alpha(\tau) \downarrow 0. \tag{1}$$

The fulfillment of (1) means that, for large τ, events determined by the beginning and the end of the process $x(t)$ become weakly dependent.

In the author' s papers ([2],[3]) the following condition of weakening of dependence between $\mathfrak{M}_{-\infty}^0$, $\mathfrak{M}_\tau^\infty$ was also used: for every $B \in \mathfrak{M}_\tau^\infty$, with probability 1,

$$\left|\mathbf{P}(B \mid \mathfrak{M}_{-\infty}^0) - \mathbf{P}(B)\right| \leq \varphi(\tau) \downarrow 0 \quad \text{as } \tau \to \infty. \tag{2}$$

In the present note a number of theorems are formulated concerning the properties of the spectral function $F(\lambda)$ of a Gaussian process $x(t)$ satisfying requirements (1) or (2). Since the spectral function $F(\lambda)$ of a Gaussian process possessing property (1) or (2) is absolutely continuous, below we shall everywhere speak not of it, but of the spectral density (s.d.) $f(\lambda) = F'(\lambda)$ of the process.

2. **Theorem 1.** *In order that a stationary Gaussian process $x(t)$ satisfy condition (2), it is necessary and sufficient that, for sufficiently large τ, $\tau > \tau_0$, the σ-algebras $\mathfrak{M}_{-\infty}^0$, $\mathfrak{M}_\tau^\infty$ be independent.*

Restating the independence condition for the σ-algebras $\mathfrak{M}_{-\infty}^0$, $\mathfrak{M}_\tau^\infty$ in spectral language, we arrive at the following variant of Theorem 1:

Theorem 1′. *In order that the function $f(\lambda)$ be the s.d. of a stationary Gaussian process satisfying condition (2), it is necessary and sufficient that it be the square of the modulus of some trigonometric polynomial—in the discrete case,* *or that it be an entire transcendental function of exponential type with exponents $\leq \sigma < \infty$, nonnegative and summable on $(-\infty, \infty)$, in the continuous-time case.*

3. **Theorem 2.** *The spectral density $f(\lambda)$ of a stationary Gaussian process $x(t)$ possessing the property of strong mixing has no discontinuities of the first kind.*

In the proof of this and the following theorems, an inequality obtained in ([4]) is essentially used,

$$\alpha(\tau) \leq \rho(\tau) \leq 2\pi\alpha(\tau),$$

$$\rho(\tau) = \sup_{\xi \in \mathfrak{M}^0_{-\infty},\, \eta \in \mathfrak{M}^\infty_\tau} \frac{\mathbf{E}(\xi - \mathbf{E}\xi)(\eta - \mathbf{E}\eta)}{\sqrt{\mathbf{E}(\xi - \mathbf{E}\xi)^2\, \mathbf{E}(\eta - \mathbf{E}\eta)^2}}, \tag{3}$$

where by $\mathfrak{M}^b_a$ is denoted the closed linear hull (in the mean-square sense) of the quantities $x(t)$, $a \leqslant t \leqslant b$.

Theorem 3. If $f(\lambda)$ is the spectral density of a stationary Gaussian process $x(t)$ having the property of strong mixing, then for all $\delta > 0$

$$\lim_{\lambda \to \lambda_0} f(\lambda)|\lambda - \lambda_0|^\delta = 0,$$

in other words, it is impossible that

$$\lim_{\lambda \to \lambda_0} f(\lambda)|\lambda - \lambda_0|^\delta = \infty.$$

Moreover, if on some set Λ

$$\lim_{\lambda \to \lambda_0,\ \lambda \in \Lambda} f(\lambda)|\lambda - \lambda_0|^\delta = \infty,$$

then

$$\lim_{\varepsilon \to 0} \frac{\operatorname{mes}(\Lambda \cap (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon))}{2\varepsilon} = 0.$$

4. Let us call, respectively, the **order**, the **upper order**, and the **lower order** of the zero λ_0 of the function $f(\lambda)$:

$$k(\lambda_0) = \lim_{\lambda \to \lambda_0} \frac{\log f(\lambda)}{\log|\lambda - \lambda_0|},$$

$$\overline{k}(\lambda_0) = \overline{\lim_{\lambda\to\lambda_0}} \frac{\log f(\lambda)}{\log|\lambda-\lambda_0|},$$

$$\underline{k}(\lambda_0) = \underline{\lim}_{\lambda\to\lambda_0} \frac{\log f(\lambda)}{\log|\lambda-\lambda_0|}. \qquad (*)$$

Theorem 4. If $f(\lambda)$ is the spectral density of a stationary Gaussian process possessing the property of strong mixing, then between the upper and lower order of the zero λ_0 of the function $f(\lambda)$ there necessarily lies an even integer

$$\underline{k}(\lambda_0) \leqslant 2n \leqslant \overline{k}(\lambda_0), \qquad n = 0, 1, \dots, \infty,$$

i.e. the true order of the zero λ_0, $k(\lambda_0)$, can only be an integer and even**.

Remark. If $x(t)$ is a stationary Gaussian process possessing the property of strong mixing, then with the aid of (3) it is easy to show that there exists an interval $(-T, T)$ such that

$$\inf_{\xi} E|x(0)-\xi|^2 = \sigma_T^2 > 0,$$

where the infimum is taken over all ξ belonging to the closed (in the mean-square sense) linear hull of the quantities $x(t)$, $t \geqslant T$.

Therefore, from the results of A. M. Yaglom ([5]) for processes with discrete time, the following additional characteristic of the zeros of their spectral density is obtained:

If $f(\lambda)$ is the spectral density of a stationary Gaussian process $x(t)$ with discrete time, $t = \dots, -1, 0, 1, \dots$, possessing the property of strong mixing, then there exists a polynomial

$$P(e^{i\lambda}) = \sum_{0\leqslant k<T} \alpha_k e^{i\lambda k},$$

* This definition was pointed out to me by A. N. Kolmogorov.
** The formulation of Theorem 4 given here belongs to A. N. Kolmogorov.

that

$$\int_{-\pi}^{\pi} \frac{|P(e^{i\lambda})|^2}{f(\lambda)}\, d\lambda < \infty.$$

I express my deep gratitude to A. N. Kolmogorov for a number of valuable comments.

SovietRxiv

Leningrad State University
named after A. A. Zhdanov

Received
19 XI 1960

References

1. M. Rosenblatt, Proc. Nat. Acad. Sci. USA, **42**, 43 (1956).

2. I. A. Ibragimov, DAN, **125**, No. 4 (1959).

3. I. A. Ibragimov, Vestn. LGU, No. 1 (1960).

4. A. N. Kolmogorov, Yu. A. Rozanov, *Theory of Probability and Its Applications*, **5**, issue 3 (1960).

5. A. M. Yaglom, UMN, **4**, issue 4 (1949).

Note: Figure translations are in progress. See original paper for figures.