Mathematics

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Abstract

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Approximation by Piecewise-Polynomial Functions

(Presented by Academician A. N. Kolmogorov, 19 VII 1961)

1. Let $D^r\Delta^s\varphi$ denote the class of functions $f(x)$, defined on the interval $[0,1]$, having there a continuous r-th derivative $f^{(r)}(x)$* ($r = 0, 1, \dots;\ f^{(0)}(x) = f(x)$), for which

$$\omega_s(f^{(r)}, h) = O[\varphi(h)] \quad (\varphi(h) \downarrow 0 \text{ together with } h).$$

Here $\omega_s(g,h)$ is the modulus of smoothness of order s of the function $g(x)$, i.e.,

$$\omega_s(g,h) = \sup_{0\le x<x+sh\le 1} \ \sup_{|t|\le h} \left|\Delta_t^s g(t)\right|, \qquad \Delta_t^s g(x) = \sum_{\nu=0}^{s} (-1)^{s-\nu} \binom{s}{\nu} g(x+\nu t).$$

Next, let $\widetilde{D}^r\Delta^s\varphi$ denote the class of periodic functions of period 2π, $f(x)$, having a continuous r-th derivative $f^{(r)}(x)$, for which

$$\omega_s(f^{(r)}, h) = O(\varphi(h)) \quad (\varphi(h) \downarrow 0 \text{ together with } h),$$

where

$$\omega_s(g,h) = \sup_{x} \ \sup_{|t|\le h} \left|\Delta_t^s g(x)\right|.$$

The classes $\widetilde{D}^r\Delta^s\varphi$ of periodic functions have been studied in considerable detail in constructive function theory from the point of view of the rate of approximation by trigonometric polynomials. For these classes, direct theorems have been established, estimating the rate of approximation of functions from these classes, and inverse theorems, which make it possible to determine, from the rate of approximation, to which of the classes $\widetilde{D}^r\Delta^s\varphi$ a function belongs.** The corresponding theory for the classes $D^r\Delta^s\varphi$ (and approximation by algebraic polynomials) began to be developed comparatively recently, and it is still not complete. In the works of A. F. Timan, V. K. Dzyadyk, and G. Freud, direct and inverse theorems were obtained for the classes $D^r\Delta^s\varphi$ with $s = 1, 2$.** In both the algebraic and the trigonometric cases, the conditions necessary for a function to belong to the class $D^r\Delta^s\varphi$ or $\widetilde{D}^r\Delta^s\varphi$, as given by the direct theorems, generally speaking, do not coincide with the sufficient conditions following from the inverse theorems. For example, for the class $\widetilde{D}^0\Delta^s\varphi$, the direct theorem asserts that if $f(x)$ belongs to this class, then for every n it can be approximated by a trigonometric polynomial of degree $\leq n$ with accuracy up to

$$C\varphi\left(\frac{1}{n}\right)$$

(C does not depend on n). The inverse theorem asserts that if $f(x)$ can be approximated for every n by a trigonometric polynomial of degree $\leq n$ with the accuracy indicated above, then it belongs to the class $\widetilde{D}^0\Delta^s\varphi_1$, where

$$\varphi_1(h) = h^s \int_h^c \frac{\varphi(t)}{t^{s+1}}\, dt.$$

Only for a few classes do $\varphi(h)$ and $\varphi_1(h)$ have the same order of decrease. It has been observed that in this respect it is more natural to use local approximations (see ([1],[3])).

* At the endpoints of the interval, $f^{(r)}(x)$ should be understood as the corresponding one-sided derivatives.

** See ([4]), Chapters 5 and 6, where a detailed bibliography is given.

The purpose of our note is to indicate linear approximation processes that more precisely characterize the classes $D^r\Delta^s_\varphi$, $\widetilde{D}^r\Delta^s_\varphi$, and the class $\bar{D}^r\Delta^s_\varphi$ of functions defined on the entire real line, subject to the same conditions as the functions in $\widetilde{D}^r\Delta^s_\varphi$, except for the periodicity condition, and also characterize certain other similar classes. The principal case is that of functions defined on a finite interval; the others can be reduced to it. A sequence of approximation processes is constructed. To describe the class $D^r\Delta^s_\varphi$, a process having number $r+s$ is used. Visually, this process may be described as "local interpolation"by algebraic polynomials of degree $\leq r+s-1$. We note that processes with number $k = 1, 2, 3$ are used in analysis to obtain quadrature formulas.

2. Let an integer $k > 0$ and a system of numbers

$$-1 \leq t_1^{(k)} < \dots < t_k^{(k)} \leq 1$$

be given. For a function $f(x)$, defined on $[0, 1]$, and a natural number n, we construct the function $P_{k,n}(f; x)$ as follows: on each interval

$$\left[\frac{\nu}{n}, \frac{\nu+1}{n}\right)$$

($\nu = 0, 1, \dots, n-1$) we set $P_{k,n}(f; x)$ equal to the algebraic polynomial of degree $\leq k-1$ interpolating $f(x)$ at the points

$$x_\mu^{(k,n,\nu)} = \frac{1}{2n}(2\nu + 1 + t_\mu^{(k)}), \qquad \mu = 1, 2, \dots, k.$$

At the point $x = 1$ we define $P_{k,n}(f;x)$ by continuity. The sequence $\{P_{k,n}(f;x)\}_{k=1}^{\infty}$ will implement an approximation process with number k. Now let $f(x)$ be a function bounded on the interval $[0,1]$. Denote

$$\tau_{k,n}(f) = \sup_{p\geq n} \sup_{0\leq x\leq 1} \left| f(x) - P_{k,p}(f;x) \right| .$$

Theorem 1. *For any function $f(x)$ bounded on the interval $[0,1]$ and fixed natural s, the relation*

$$\omega_s\left(f;\frac{1}{n}\right) \asymp \tau_{s,n}(f)$$

holds.

In proving the theorem, the following inequality, due to Whitney (^{5}), is used:

$$E_{s-1}(f;a,b) = \inf_{c_k} \sup_{a\leq x\leq b} \left| f(x) - \sum_{k=0}^{s-1} c_{kx}^{k} \right| \leq A_s \omega_s\left(f;\frac{b-a}{s}\right),$$

where A_s depends only on s, and $f(x)$ is bounded on $[a,b]$.

Theorem 2. *For any function $f(x)$ bounded on the interval $[0,1]$ and fixed natural r, one has*

$$\omega_s\left(f;\frac{1}{n}\right) = O\left(\frac{1}{n^s}\sum_{k=1}^{n} k^{s-1}\tau_{r,k}(f)\right), \qquad s = 1,2,\ldots,r-1;$$

but if $f(x)$ is such that, for some natural $\nu < r$, the series

$$\sum_{k=1}^{\infty} k^{\nu-1}\tau_{r,k}(f)$$

converges, then $f(x)$ has a continuous derivative $f^{(\nu)}(x)$ of order ν, and moreover

$$\omega_{r-\nu-j}\left(f^{(\nu)};\frac{1}{n}\right) = O\left(\sum_{k=n+1}^{\infty} k^{\nu-1}\tau_{r,k}(f) + \frac{j}{n^{r-\nu-j}}\sum_{k=1}^{n} k^{r-j-1}\tau_{r,k}(f)\right)$$

$$(0 \leq j < r-\nu).$$

Theorems 1 and 2 in a number of cases (for example, for the classes $D^0\Delta^s\varphi$, as well as $D^r\Delta^s t^{\alpha}$, $0 < \alpha \leq s$) give necessary and sufficient conditions for membership in a class. Without dwelling on this in detail, let us note that analogous theorems can be obtained for periodic functions, and also for functions defined on a set consisting of several finite or infinite intervals of the real axis.

Corollary 1 (A. Marchaud). If $s \leq r$, $|f(x)| \leq M$, then

$$\omega_s(f;h) = O\left(h^s \int_h^{1/r} \frac{\omega_r(f;t)}{t^{s+1}}\,dt + Mh^s\right), \qquad h \leq \frac{1}{2r}.$$

This relation was obtained directly by Marchaud (^{2}).

Corollary 2. If, for some natural number $\nu \le r-1$, the integral

$$\int_0^{1/r} \frac{\omega_r(f;t)}{t^{\nu+1}}\,dt$$

converges and $|f(x)| \le M$, then $f(x)$ has on $[0,1]$ a ν-th continuous derivative, and

$$\omega_{r-\nu-j}(f^{(\nu)},h) = O\left(\int_0^h \frac{\omega_r(f;t)}{t^{\nu+1}}\,dt + h^{r-\nu-j}\int_h^{1/r} \frac{\omega_r(f;t)}{t^{r-j+1}}\,dt + Mh^{r-\nu+j}\right); \tag{1}$$

$$\omega_{r-\nu}(f^{(\nu)},h) = O\left(\int_0^h \frac{\omega_r(f;t)}{t^{\nu+1}}\,dt\right), \qquad 1 \le j < r-\nu, \quad h \ge \frac{1}{2r}. \tag{2}$$

Until now relation (2) was known only for the case $\nu = r-1$ (it was obtained by Marchaud in (^{2})), and relation (1) only for continuous functions defined on the whole axis (it follows from the direct and inverse theorems for approximation by entire functions). It is easy to see that relations (1) and (2) carry over to the case of functions defined on a set consisting of several finite or infinite intervals, if one assumes that these functions grow as $O(|x|^r)$ as $|x| \to \infty$.

It follows from (2), for example, that if $\omega_{r+1}(f;h) = O(h^{r+\alpha})$ $(0 < \alpha \le 1)$, then $f(x)$ has on $[0,1]$ an r-th continuous derivative satisfying a Hölder condition of order α, while if $\omega_{r+1}(f;h) = O(h^r)$, then $f(x)$ has on $[0,1]$ an $(r-1)$-st continuous derivative satisfying a Zygmund-type smoothness condition, etc.

Approximation by means of $P_{k,n}(f;x)$ is closely connected with the local best approximation $E_{k-1}(f;c,d)$, $(c,d) \subset (0,1)$. Namely, the following is true:

Theorem 3. Let $f(x)$ be a bounded function on $[0,1]$, and let k be a fixed natural number; then

$$\sup_{0\le x\le 1-\frac{1}{n}} E_{k-1}\left(f;x,x+\frac{1}{n}\right) \asymp \tau_{k,n}(f).$$

Let us note that from Theorems 1-3 one obtains, for example, the following theorem (D. A. Raikov (^{3})):

A function $f(x)$, continuous on $[a,b]$, has a derivative $f^{(r)}(x)$ of order r, satisfying a Hölder condition of order α $(0 < \alpha \le 1)$, if and only if for any $a \le c < d \le b$ the inequality

$$E_r(f;c,d)=O\big[(d-c)^{r+\alpha}\big]$$

holds.

3. If one restricts attention to the classes $D^r\Delta^s t^s$, then Theorems 1 and 2 admit a further refinement. Namely, as $t_\nu^{(k)}$, $\nu=1,2,\dots,k$,

in constructing the k-th approximation process we take the numbers

$$t_\nu^{(k)}=\cos\frac{2k-2\nu+1}{2k}\pi.$$

Theorem 4. In order that a function $f(x)$ bounded on $[0,1]$ have there an r-th derivative $f^{(r)}(x)$ satisfying the condition $\omega_s(f^{(r)};h)\le Mh^s$, it is necessary and sufficient that for all natural n

$$\tau_{r+s,n}(f)\le\frac{M}{(r+s)!\,2^{2r+2s-1}}\frac{1}{n^{r+s}}. \tag{3}$$

Remark. In Theorem 4 the approximation is carried out with the aid of discontinuous functions. If, as $t_\nu^{(k)}$, one takes the numbers

$$t_\nu^{(k)}=\frac{\cos\frac{2k-2\nu+1}{2k}\pi}{\cos\frac{\pi}{2k}},$$

then the corresponding $P_{k,n}(f;x)$ will already be continuous. In this case condition (3) of Theorem 4 is replaced by the condition

$$\tau_{r+s,n}(f)\le\frac{M}{(r+s)!\,2^{2r+2s-1}\left(\cos\frac{\pi}{2r+2s}\right)^{r+s}}\frac{1}{n^{r+s}},$$

which is also necessary and sufficient.

From Theorem 4 we obtain

Corollary 1. If $f(x)$ is a function bounded on $[0,1]$ and r is natural, then all the conditions

$$\omega_{r-j}(f^{(j)},t)\le Mt^{r-j},\qquad j=0,1,\dots,r-1,$$

are equivalent.

Under somewhat different assumptions this result was obtained earlier by D. A. Raikov (^{3}).

Corollary 2. In order that a function $f(x)$ given on $[0,1]$ be infinitely differentiable with $\max_{0\le x\le 1}|f^{(r)}(x)| = M_r$, where $\{M_r\}_{r=1}^{\infty}$ is some sequence of positive numbers, it is necessary and sufficient that for all natural r and n

$$T_{r,n}(f) \le \frac{M_r}{r!\,2^{2r-1}}\frac{1}{n^r}.$$

With the aid of Corollary 2 one can give a constructive characteristic of certain classes of infinitely differentiable functions. We shall restrict ourselves to considering Gevrey classes. A function $f(x)$ belongs to the Gevrey class of order $\alpha > 0$ if it is infinitely differentiable on $[0,1]$ and, for some $A > 0$,

$$\max_{0\le x\le 1}|f^{(r)}(x)| \le A^r r^{\alpha r}, \qquad r = 1, 2, \ldots$$

Theorem 5. In order that a function $f(x)$, given on $[0,1]$, belong to the Gevrey class of order $\alpha > 0$, it is necessary and sufficient that, for some $A_1 > 0$,

$$\tau_{r,n}(f) = O\left(\exp\left\{-[A_1 rn]^{1/\alpha}\right\}\right).$$

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Note: Figure translations are in progress. See original paper for figures.