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Abstract

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ON THE GENERAL FORM OF A FUNCTIONAL IN THE SPACE OF FUNCTIONS ANALYTIC IN A SEMICIRCULAR DOMAIN

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Let T be some domain in the space of the complex variables z, w. Denote by A_T the space of functions $f(z, w)$ analytic in the domain T. Convergence of a sequence of elements of this space is defined as uniform convergence in every domain lying strictly inside T. If the domain T is the bicylinder $\{|z| < R_1,\ |w| < R_2\}$, then, as is known, any linear functional G in the space A_T is defined by the formula

$$G(f) = \int_{|z|=R_1-\varepsilon} \int_{|w|=R_2-\varepsilon} f(z,w) g(z,w)\, dz\, dw \tag{1}$$

with a function $g(z, w)$ analytic in the domain $\{|z| > R_1 - 2\varepsilon,\ |w| > R_2 - 2\varepsilon\}$, where $\varepsilon > 0$ and depends on G. This correspondence between the functionals G and the functions $g(z, w)$ becomes one-to-one if one requires that $g(\infty, \infty) = 0$.

In the case when T is a complete n-circular domain, the general form of a functional in the space A_T was obtained by S. D. Okun* and L. A. Aizenberg and B. S. Mityagin ([1]). In this note we consider the space of functions analytic in a semicircular domain**.

Let T be a complete semicircular domain with plane of symmetry $w = 0$. By definition of a semicircular domain,

$$T = \{z \in H_T;\ |w| < R_T(z)\},$$

where H_T is some domain in the z-plane, and $R_T(z)$ is some nonnegative function defined in the domain H_T. As is known, every function $f(z, w) \in A_T$ expands into a Hartogs series

$$f(z, w) = \sum_{k=0}^{\infty} w^k f_k(z).$$

Here the functions $f_k(z)$, analytic in the domain H_T, satisfy the condition

$$\overline{\lim_{z'\to z}}\ \overline{\lim_{k\to\infty}} \sqrt[k]{|f_k(z')|} \le \frac{1}{R_T(z)}.$$

Considering the bicylinder $\{|z| < R_1,\ |w| < R_2\}$ as a semicircular domain, we shall give formula (1) a somewhat different form, more convenient for us.

* The results of S. D. Okun were reported at the Fifth All-Union Conference on Function Theory in Yerevan.

** On semicircular domains see, for example, ([2]).

For this purpose, let us expand the functions $f(z,w)$ and $g(z,w)$ in Hartogs series

$$f(z,w) = \sum_{k=0}^{\infty} w^k f_k(z), \qquad g(z,w) = \sum_{k=0}^{\infty} w^{-k-1} g_k(z).$$

Since the functions $f(z,w)$ and $g(z,w)$ are holomorphic respectively in the bicylinders $\{|z| < R_1,\ |w| < R_2\}$ and $\{|z| > R_1 - 2\varepsilon,\ |w| > R_2 - 2\varepsilon\}$, the functions $f_k(z)$ and $g_k(z)$ are analytic, respectively, in the domains $(|z| < R_1)$ and $(|z| > R_1 - 2\varepsilon)$, and for $|z| = R_1 - \varepsilon$ satisfy the conditions

$$\overline{\lim}_{z'\to z}\ \overline{\lim}_{k\to\infty} \sqrt[k]{|f_k(z')|} < \frac{1}{R_2}, \qquad \overline{\lim}_{z'\to z}\ \overline{\lim}_{k\to\infty} \sqrt[k]{|g_k(z')|} < R_2 - 2\varepsilon. \quad (2)$$

Perform in formula (1) the integration with respect to w:

$$\begin{aligned} G(f) &= \int_{|z|=R_1-\varepsilon} \int_{|w|=R_2-\varepsilon} f(z,w) g(z,w)\, dz\, dw \\ &= \int_{|z|=R_1-\varepsilon} \int_{|w|=R_2-\varepsilon} \left(\sum_{k=0}^{\infty} w^k f_k(z) \right) \left(\sum_{k=0}^{\infty} w^{-k-1} g_k(z) \right) dz\, dw \\ &= \int_{|z|=R_1-\varepsilon} \sum_{k=0}^{\infty} f_k(z) g_k(z)\, dz = \sum_{k=0}^{\infty} \int_{|z|=R_1-\varepsilon} f_k(z) g_k(z)\, dz. \end{aligned}$$

Thus, a linear functional in the space under consideration can be specified by a sequence of functions $g_k(z)$, analytic for $|z| > R_1 - 2\varepsilon$ and satisfying, on some contour lying strictly inside $H_T = \{|z| < R_1\}$, the condition

$$\overline{\lim}_{z'\to z}\ \overline{\lim}_{k\to\infty} \sqrt[k]{|g_k(z')|} < R_T(z) - \varepsilon,$$

where $R_T(z) = R_2$. Obviously, consideration of bicylinders whose center does not lie at the origin introduces nothing new in comparison with the case considered.

In the case of a semicircular domain the following holds:

Theorem. Let G be a continuous linear functional in the space A_T, where T is a complete semicircular domain. Then there exists a contour C, lying strictly inside H_T, and a sequence of functions $g_k(z)$, analytic outside the contour C and on the contour C itself, such that for every function $f(z,w) \in A_T$

$$G(f) = \sum_{k=0}^{\infty} \int_C f_k(z) g_k(z)\, dz. \tag{3}$$

Moreover, a linear functional in A_T is determined by the sequence $\{g_k(z)\}$ if and only if the sequence under consideration can be represented in the form

$$g_k(z) = \sum_{i=1}^{N} g_{k,i}(z),$$

where N depends on G and does not depend on k, and the functions $g_{k,i}(z)$ satisfy the conditions: 1) for every k, the function $g_{k,i}(z)$ must be holomorphic outside some contour C_i, lying inside H_T, and also on the contour itself; 2) for every $z \in C_i$ the condition must hold

$$\overline{\lim}_{z' \to z}\ \overline{\lim}_{k \to \infty} \sqrt[k]{|g_{k,i}(z')|} < R_T(z) - \varepsilon.$$

The representation is unique under the additional requirement $g_k(\infty) = 0$, $k = 1, 2, \ldots$.

Proof. Take a sequence of finite semicircular domains $\{T_p\}$ such that $T_p \subset T_{p+1}$ for every p and $T_p \to T$ as $p \to \infty$. Denote by $\dot{T}$ the boundary of the domain T. Also put

$$\varepsilon_p = \inf_{(z,w)\in\dot{T};\ (z',w')\in T_p} \sqrt{|z'-z|^2 + |w-w'|^2},$$

and require that $\varepsilon_p > \varepsilon_{p+1} > 0$ for every p. It is obvious that such a choice of the sequence $\{T_p\}$ is possible for any semicircular domain T. With the aid of the domains T_p, introduce in the space A_T a countable sequence of norms

$$\|f\|_p = \max_{(z,w)\in T_p} |f(z,w)|.$$

The topology defined by this set of norms is, clearly, equivalent to the topology of the space A_T introduced earlier.

Let G be a linear continuous functional in the space A_T. As is known ([3]), a functional continuous in a countably normed space is continuous with respect to one of the norms of this space. Consequently, the functional G will be continuous with respect to some norm $\| \, \|_q$. Take in the domain T_q an arbitrary disk $\{|w| < R_T(z_0);\ z = z_0 \in H_{T_q}\}$ and cover it by the bicylinder

$$K(z_0) = \left\{ |z - z_0| < \frac{\varepsilon_q}{2\sqrt{2}};\ |w| < R_{T_q}(z_0) + \frac{\varepsilon_q}{2\sqrt{2}} \right\}.$$

Note that

$$\begin{aligned} &\inf_{(z,w)\in \dot{T}} \inf_{(z',w')\in K(z_0)} \sqrt{|z-z'|^2 + |w-w'|^2} \geq \\ &\geq \inf_{(z,w)\in \dot{T}} \inf_{(z_0,w'')\in T_q} \sqrt{|z-z_0|^2 + |w-w''|^2} - \\ &\quad - \sup_{\substack{(z',w')\in K(z_0) \\ |w''|<R_{T_q}(z_0)}} \inf \sqrt{|z'-z_0|^2 + |w'-w''|^2} \geq \varepsilon_q - \frac{1}{2}\varepsilon_q = \frac{1}{2}\varepsilon_q, \end{aligned}$$

i.e., for $z_0 \in H_{T_q}$ each bicylinder $K(z_0)$ lies strictly inside T. The totality of all bicylinders $K(z)$, $z \in H_{T_q}$, covers not only the domain T_q, but also its closure $\overline{T}_q$. Consequently, one can choose a finite number N of bicylinders $K_i = K(z_i)$, $i = 1, 2, \dots, N$, so that

$$\overline{T}_q \subset K = \bigcup K_i \subset T.$$

Since the functional G is continuous with respect to the norm $\| \, \|_q$, and $T_q \subset K$, the functional under consideration will also be continuous in the topology of the space A_K. Since $K = \bigcup K_i$, we have $A_K = \bigcap A_{K_i}$. Hence, by virtue of a proposition of V. P. Khavin (([4]), p. 239), it follows that the functional G is representable in the form

$$G = \sum_{i=1}^{N} G_i,$$

where each functional G_i is defined in the space A_{K_i} and is continuous in the topology of this space. From the remarks made at the beginning of the note it follows that to the functional G_i there corresponds a sequence of functions $\{g_{k,i}(z)\}_K$, analytic in the domain

$$|z - z_i| > \frac{\varepsilon_q}{2\sqrt{2}} - 2\varepsilon > 0,$$

such that, for $f(z, w) \in A_{K_i}$ and $C_i = |z - z_i| = \varepsilon_q/2\sqrt{2} - \varepsilon$,

$$G_i(f) = \sum_{k=0}^{\infty} \int_{C_i} f_k(z) g_k(z)\, dz.$$

Since the distance between any boundary points of the domains T and A_{K_i} is greater than $\frac{1}{2}\varepsilon_q$, we have $R_{K_i}(z) < R_T(z) - \varepsilon_q/2$ for every $z \in H_{h_i}$. Hence, from (2) it follows that for $z \in C_i$ the sequence $g_{k,i}(z)$ satisfies the condition

$$\overline{\lim_{z' \to z}}\ \overline{\lim_{k \to \infty}} \sqrt[k]{|g_{k,i}(z')|} < R_T(z) - \frac{1}{2}\varepsilon_q.$$

Now let $f(z, w) \in A_T$. Then the corresponding functions $f_k(z)$ will be analytic in the domain H_T, containing all the disks H_{K_i}. Take inside the domain H_T a contour C such that every disk $|z - z_i| < \varepsilon_q/2\sqrt{2}$ lies strictly inside C. Then, for all k and i,

$$\int_{C_i} f_k(z) g_k(z)\, dz = \int_C f_k(z) g_k(z)\, dz,$$

and, consequently,

$$G(f) = \sum_{i=1}^{N} G_i(f) = \sum_{i=1}^{N} \sum_{k=0}^{\infty} \int_{C_i} f_k(z) g_{k,i}(z)\, dz = \sum_{k=0}^{\infty} \int_C f_k(z) g_k(z)\, dz.$$

The necessity is proved.

The sufficiency is almost obvious. Indeed, if $f(z, w) \in A_T$ and the functions $g_{k,i}(z)$ satisfy the conditions of the theorem, then, as follows from the properties of functions analytic in a semicircular domain, for every $\eta > 0$, beginning with some $k = k_0(\eta)$, the inequalities

$$\sqrt[k]{|f_k(z)|} < \frac{1+\eta}{R_T(z)}; \qquad \sqrt[k]{|g_{k,i}(z)|} < R_T(z) - \varepsilon + \eta \quad (z \in C_i)$$

will hold. But then the series

$$\sum_{k=0}^{\infty} f_k(z) g_{k,i}(z)$$

converge absolutely and uniformly on the corresponding contours C_i, and, consequently, the functional defined by the formula

$$G(f) = \sum_{k=0}^{\infty} \int_C f_k(z) g_k(z)\, dz = \sum_{i=1}^{N} \sum_{k=0}^{\infty} \int_{C_i} f_k(z) g_{k,i}(z)\, dz$$

is defined on all functions $f(z, w) \in A_T$ and is continuous.

Finally, to prove the uniqueness of the representation (3), it is enough to consider $G(w^k f_k(z))$, using the uniqueness of the corresponding representation of a functional in the case of one variable.

In conclusion we note that the result obtained is transferred without difficulty to the case of a number of variables greater than two. In this case, instead of semicircular domains, one considers domains that are domains of uniform convergence of series of the form

$$\sum_{k=0}^{\infty} w^k f_k(z_1, \dots, z_n).$$

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References

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[4] V. P. Khavin, *DAN*, 121, No. 2 (1958).

Note: Figure translations are in progress. See original paper for figures.