NONSTATIONARY PROPAGATION OF WAVES IN AN INHOMOGENEOUS MEDIUM DURING THE FORMATION OF A REGION OF GEOMETRICAL SHADOW

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Abstract

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NONSTATIONARY PROPAGATION OF WAVES IN AN INHOMOGENEOUS MEDIUM DURING THE FORMATION OF A REGION OF GEOMETRICAL SHADOW

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In the present article, by means of the method of separating out the nonanalytic part ([1,2]), the properties of a nonstationary wave field are investigated in the region of geometrical shadow*, arising in an inhomogeneous half-space.

1. Let, in the cylindrical coordinate system r, φ, z, the half-space $z > 0$ be given, with a variable velocity $n^{-1}(z)$ of wave propagation, regular and monotone for $z > 0$. We shall assume that the wave field $u(r, z, t)$ is generated by the action, at $r = z = 0$, of a point source whose intensity dependence on time t is determined by the Heaviside unit function $\varepsilon(t)$, and is the solution of the problem**

$$u_{rr} + \frac{1}{r}u_r + u_{zz} - n^2(z)u_{tt} = 0,$$
$$u\big|_{t=0} = u_t\big|_{t=0} = 0, \qquad u\big|_{z=0} = r^{-1}\delta(r)\varepsilon(t). \tag{1}$$

The exact solution of problem (1) has the form

$$u(r, z, t) = \frac{1}{4\pi i}\int_0^\infty J_0(kr)\, dk \int_\lambda \frac{G(z, k, s)}{G(0, k, s)}\, \frac{e^{kst}}{s}\, ds. \tag{2}$$

In equality (2), $J_0(z)$ is the Bessel function; λ is the Mellin contour; $G(z, k, s)$ is the solution of the equation

$$\frac{d^2V}{dz^2} - k^2\left[1 + s^2n^2(z)\right]V = 0,$$

which, in the region $\mathrm{Re}(ks) \gg 0$, $|s - in^{-1}(z)| \gg |k|^{-2/3}$, has the asymptotic representation

$$G(z, k, s) = \tag{3}$$

$$= \sqrt{\frac{2}{k\pi}}\,[1+s^2n^2(z)]^{-1/4}\exp\left[-k\int\sqrt{1+s^2n^2(z)}\,dz\right]\left[1+O\left(\frac{1}{ks}\right)\right].$$

2. If $n'(z) > 0$ for all $z > 0$, then in the half-space there arises an infinite zone of geometrical shadow

$$\delta \equiv r - n_0\int_0^z \frac{dz}{\sqrt{n^2(z)-n_0^2}} > 0, \qquad n_0 \equiv n(0), \tag{4}$$

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* In the illuminated part of the inhomogeneous half-space, the nonstationary wave field for a particular law of variation of the velocity was studied in ([3]). In the more general case the field may be investigated by the ray method ([4]), or by applying the asymptotics and the method of passing to formula (2) of the present work.

** Cases in which the source is located inside the half-space, or in which the boundary condition has the form $u_z|_{z=0} = r^{-1}\delta(r)\varepsilon(t)$, may be considered analogously.

according to which, in accordance with the equation

$$\gamma = \frac{t}{n_0} - \delta - \frac{1}{n_0}\int_0^z \frac{n^2(z)}{\sqrt{n^2(z)-n_0^2}}\,dz = 0 \tag{5}$$

the sliding front (2) propagates. If, however, the function $n(z)$ increases only on the interval $0 < z < z_1$, and for $z > z_1$ decreases or oscillates, then several sliding fronts are formed in the half-space, and caustics of the sliding front also arise.

3. For $n(z)$ regular and monotonically increasing for $z > 0$, the function $G(z,k,s)$ has the following properties:

I. $G(z,k,s)$ is a regular function of s outside the cut between the points $s = \pm i n^{-1}(z)$ for fixed k and $z > 0$, and a regular function of k outside the cut for $k < 0$, if s and $z > 0$ are fixed.

II. The ratio $G(z,k,s)G^{-1}(0,k,s)$ is real for $k, s > 0$; $z \geq 0$.

III. In the domain $|s - in^{-1}(z)| < A|k|^{-2/3}$ for $|k| \gg 1$, the Langer-Fock asymptotic formula ([5,6]) is valid

$$G(z,k,s) = e^{-i\pi/6}\sqrt[4]{\frac{4p}{k^2\pi^2[1+s^2n^2(z)]}}\left[w\big(pe^{4\pi i/3}\big) + O\big(k^{-2/3}\big)\right],$$

where

$$p = \left[\frac{3k}{2} \int_{z_0}^{z} \sqrt{1 + s^2 n^2(z)}\, dz \right]^{2/3}, \qquad n(z_0) = i s^{-1},$$

and $w(t)$ is the Airy function.

IV. For the roots $s_m(k)$, $m = 1, 2, ...$, of the equation $G(0, k, s) = 0$, for $k \gg 1$ and $n'(0) = n'_0 > 0$, the asymptotic formula is valid

$$s_m(k) = i n_0^{-1} + n_0^{-5/3} ({n'}_0 k^{-1})^{2/3} x_m \exp\left(\frac{7\pi i}{6}\right) + O\big(k^{-4/3}\big)\,,$$

$$m = 1, 2, ... \ll k,$$

in which x_m is a root of the function $w(2^{1/3} x e^{i\pi/3})$. The roots $s_m(k)$ are simple, at least for sufficiently large values of $|k|$. $s_m(k)$ is a regular function of k for large $|k|$.

V. If s_1 and s_2 are distinct roots of the equation $G(z, k, s) = 0$, then the orthogonality relation holds

$$\int_{z}^{\infty} n^2(x) G(x, k, s_1) G(x, k, s_2)\, dx = 0 \qquad (\mathrm{Re}\, k \gg 0).$$

VI. For real x, $\delta > 0$ and $z \geq 0$, the function $G(z, -ix, i\delta)$ is real.

4. Let us transform $u(r, z, t)$ in a neighborhood of the sliding front $\gamma = 0$. Changing the lower limit $k = 0$ to $k = k_0 \gg 1$ affects only the part of the function $u(r, z, t)$ that is regular at $\gamma = 0$. Using property II, we replace the inner integral over s by the real part of the integral continued over a contour belonging to the upper half-plane, and then compute it by residues at the roots $s_m(k)$. We interchange the sum over residues with the integral over k and use the possibility of changing the lower limit of integration. Using properties I, IV, V, VI and rotating, for $\gamma < 0$, the contour of integration in the k-plane through an angle $(-\pi/2)$, one can show that the isolated analytic part of the field in a neighborhood of the sliding front is identically zero. This makes it possible, after simple transforma-

···we arrive at the final formula

$$u(r, z, t) = \frac{3i}{4} \sum_{m=1}^{\infty} \int_{\infty e^{-i\pi/3}}^{\infty e^{i\pi/3}} H_0^{(2)}(-i\xi^3 r)\, G(z, i\xi^3, i\sigma_m) \times$$

$$\times \left[\frac{\partial G(0, i\xi^3, s)}{\partial s} \right]_{s = i\sigma_m}^{-1} e^{-\sigma_m \xi^3 t}\, \frac{\xi^2\, d\xi}{\sigma_m}\, \varepsilon(\gamma), \tag{6}$$

valid in a finite region enclosing the surface $\gamma = 0$. In (6), σ_m is the m-th root of the equation $G(0, i\xi^3, i\sigma) = 0$, real, according to V and VI, for real ξ; $H_0^{(2)}(z)$ is the Hankel function.

5. From the basic formula (6), simple asymptotic expressions may be obtained for the field in the neighborhood of the sliding front.

Let $n_0' \neq 0$. Using the properties (3), III and IV, and applying the saddle-point method, we arrive at the asymptotic equality

$$u(r,z,t) = \sqrt{\frac{n_0}{r}} \left(\frac{n_0'}{n_0}\right)^{5/6} \frac{(6\gamma)^{1/4}\, |\omega'(2^{1/3}\chi_1 e^{i\pi/3})|^{-1}}{(\chi_1\delta)^{5/4}\, \sqrt[4]{n^2(z) - n_0^2}} \times$$
$$\times \exp\left[-\frac{2}{3}\frac{(\chi_1\delta)^{3/2}}{\sqrt{3\gamma}}\right][1 + O(\sqrt{\gamma})]\,\varepsilon(\gamma). \tag{7}$$

If $\gamma \to 0$, the influence of the subsequent terms of the series in (6) (for $m = 2, 3, \ldots$) is not significant in comparison with the corrections to the principal term ($m = 1$). Formula (7) becomes inapplicable when approaching the limiting ray ($\delta \to 0$) and the boundary of the half-space ($z \to 0$). In the region $z \ll \gamma$, adjacent to the boundary of the half-space, one can obtain a formula analogous to (7).

In the more general case, when the quantity n_0' may be equal to zero, in carrying out the saddle-point method one should use, for the functions $G(z, k, s)$, the asymptotic formulas (7). Suppose that $n(z)$ in the neighborhood of $z = 0$ behaves as

$$n_0 + az^\alpha \qquad (\alpha, a > 0).$$

In this case the behavior of the field in the neighborhood of the sliding front is determined by the factor

$$\exp\left[-A_0(\gamma+\delta)^{\frac{2+\alpha}{2\alpha}}\gamma^{-\frac{2-\alpha}{2\alpha}}\right], \qquad A_0 > 0. \tag{8}$$

For $\alpha = 1$, (8) agrees with (7). For $\alpha \geq 2$ the singularity of the field characteristic of the sliding front (continuity of the field together with all derivatives in passing through the sliding front) disappears, and a singularity inherent in ordinary fronts appears. This agrees with formulas (4), (5), from which it follows that for $\alpha \geq 2$ the region of geometrical shadow disappears. For $0 < \alpha < 2$, expression (8) characterizes the change of the singularity on the sliding front with the change of the index of refraction.

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CITED LITERATURE

1. V. S. Buldyrev, DAN, **129**, No. 2, 291 (1959).
2. V. S. Buldyrev, I. A. Molotkov, DAN, **134**, No. 5, 1051 (1960).
3. A. S. Alekseev, Uchen. zap. LGU, No. 246, 167 (1958).
4. V. M. Babich, A. S. Alekseev, Izv. AN SSSR, ser. geofiz., No. 1, 17 (1958).
5. R. Langer, Phys. Rev., **51**, No. 8, 669 (1937).
6. V. A. Fok, *Tables of Airy Functions*, 1946.
7. A. A. Dorodnitsyn, UMN, **7**, No. 6, 3 (1952).

Note: Figure translations are in progress. See original paper for figures.