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Abstract

Full Text

Mathematics

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The Binary Case of an Additive Problem with Squares of Prime Numbers

(Presented by Academician I. M. Vinogradov on May 10, 1961)

1. The problem of representing numbers as a sum of s squares of prime numbers for any $s \geq 5$ was solved by I. M. Vinogradov ⁽¹⁾. By the same method, in ⁽²⁾ the question was studied for $s = 3, 4$.

In the present note a solution is given for the last case ($s = 2$) of the problem with squares of primes. Earlier, for the principal quantities characterizing the additive properties of the sequence $p^2 + q^2$ (p, q prime), P. Erdős ⁽³⁾ obtained results of the inequality type.

Theorem 1. The number $F(x, D)$ of numbers $n \leq x$ representable in the form

$$n = p^2 + q^2, \quad (1)$$

with primes p and q from the progression $Dk + l$; $(D, l) = 1$; $D \leq (\ln x)^M$, where M is an arbitrarily prescribed positive constant, is expressed by the formula

$$F(x, D) = \frac{\pi}{2\varphi^2(D)} \frac{x}{\ln^2 x} \left(1 + O\left(\frac{1}{\ln^{1-\varepsilon} x}\right) \right),$$

where φ is Euler's function; $\varepsilon > 0$ is an arbitrarily small quantity.

Theorem 2. The number $T(x, D)$ of numbers n in the interval $(0, x)$ for which equation (1) has two or more solutions*, satisfies the estimate

$$T(x, D) \ll \frac{x}{\varphi^2(D) \ln^{3-\varepsilon} x}.$$

An immediate consequence of these theorems is, in particular, the following:

Theorem 3. Asymptotically all numbers representable by the form $p^2 + q^2$ with primes p and q are represented by it in a unique way.

The form $p^2 - q^2$ (p, q prime numbers) has entirely analogous additive properties.

Let, further, l_k for $k = 1, 2, \dots$ denote the line with slope one, intersecting the axis of abscissas at the point $(2k, 0)$.

Theorem 4. The number $I_k(x)$ of prime points (p, q) at the intersection of the line l_k with the concentric circles $p^2 + q^2 = n$, $n \leq x$, is expressed by the formula

$$I_k(x) = 4\sqrt{2} \prod_{g>2} \frac{g(g-2)}{(g-1)^2} \frac{\sqrt{x}}{\ln^2 x} \prod_{\substack{g>2 \\ g|k}} \frac{g-1}{g-2} \left(1 + O\left(\frac{1}{\ln x}\right) \right)$$

for every $2 \leq 2k \leq \sqrt{x}/\ln x$, excluding no more than

$$\ll \sqrt{x}(\ln x)^{-M}$$

* Solutions differing by the order of the summands are not counted as distinct. of them, where $M > 0$ is any prescribed constant; g runs through the prime numbers; \ll, O are quantities depending only on M .

The principal basis for deriving Theorems 1-3 is the “sieve” method. Theorem 4 is a consequence of the method of trigonometric sums of I. M. Vinogradov. (With the help of the “sieve,” with regard to the quantity $I_k(x)$, at present one can obtain only upper estimates.)

2. We introduce additional notation: p, q, ξ, η are prime numbers of the progression $Dk + l$, $(D, l) = 1$, $l \leq D$; $D \leq L^\alpha$, $L = \ln x$; $\pi(z, D, l)$ is the number of primes $p \leq z$; $\vartheta(z, D, l) = \sum'_{p \leq z} \ln p$, B is a bounded quantity.

For a given integer n , consider the system of equations

$$\begin{aligned} n &= p^2 + q^2, & p &> q, \\ n &= \xi^2 + \eta^2, & \xi &> \eta. \end{aligned} \tag{2}$$

We represent the number $N^2(n)$ of solutions of (2) in the form

$$N^2(n) = N_1(n) + 2N_2(n), \tag{3}$$

where $N_2(n)$ denotes the number of solutions of (2) with $p > \xi$. Then, summing (3) over all n not exceeding x , we shall have

$$\sum_{n \leq x} N^2(n) = Q(x) + BQ'(x), \tag{4}$$

where $Q(x)$ is the number of solutions of the inequality $p^2 + q^2 \leq x$, $p > q$, and

$$Q'(x) = \sum_{0 < m \leq x} \sum_{\substack{q, \eta, \xi, p \leq \sqrt{x} \\ q < \eta < \xi < p}} \sum_{m=p^2-\xi^2} \sum_{m=\eta^2-q^2} 1. \quad (5)$$

Put

$$T_\nu(x) = \sum_{h=0}^{\nu} \frac{(-1)^h}{(h+1)!} \sum_{n \leq x} \prod_{s=0}^h \{N(n) - s\}. \quad (6)$$

Then for any integers $t \geq 0$, $\nu \geq 0$, the number $F(x, D)$ of numbers $\leq x$ representable in the form $p^2 + q^2$, $p > q$, will satisfy the relations

$$T_{2t+1}(x) \leq F(x, D) \leq T_{2\nu}(x). \quad (7)$$

Moreover, for the number $G(x, D)$ of numbers $\leq x$ representable in the form $p^2 + q^2$ with $p > q$ in only one way, the inequalities

$$T_1(x) \leq G(x, D) \leq Q(x) \quad (8)$$

hold.

The derivation of relations (7) and (8) repeats the arguments given by us in the note ⁶.

From (6)-(8) it follows that

$$\frac{3}{2}Q(x) - \frac{1}{2} \sum_{n \leq x} N^2(n) \leq G(x, D) \leq F(x, D) \leq Q(x), \quad (9)$$

and from comparing (9) and (4) we conclude that

$$\left. \begin{aligned} F(x, D) - Q(x) \\ G(x, D) - Q(x) \end{aligned} \right\} = BQ'(x). \quad (10)$$

3. We next estimate the quantities in (10). For $Q(x)$, using the law of distribution of prime numbers in progressions, we obtain an asymptotic expression, while $Q'(x)$ is estimated from above by the "sieve" in the form of the following lemma:

Lemma. The number $W(z; u_1, \dots, u_m)$ of primes $p = Dt + l \leq z$ such that, for distinct integers u_1, \dots, u_m , $u_i \neq 0$ ($i = 1, \dots, m$), the numbers

$$p + u_1, \dots, p + u_m$$

are also prime and belong to the progression $Dt + l$ with $l \leq D$, $D \leq (\ln z)^M$, satisfies the estimate

$$W(z; u_1, \dots, u_m) \ll \frac{z}{D \ln^{m+1} z} \prod_g \frac{g - \gamma(g, D)}{g \left(1 - \frac{1}{g}\right)^{m+1}} + B, \quad ()$$

where the product is extended over all primes g ; $\gamma(g, D)$ is the number of solutions of the congruence

$$(Dt + l)(Dt + l + u_1) \cdots (Dt + l + u_m) \equiv 0 \pmod{g}.$$

The proof of this lemma is contained in the work of N. I. Klimov (5).

Estimate of $Q'(x)$. Put

$$p - \xi = 2u, \quad \eta - q = 2v \quad (11)$$

and $u = u'd$, $v = v'd$, where u' , v' are coprime integers. We find that

$$\xi = v' \frac{g + v}{u'} - u,$$

whence

$$q \equiv -v \pmod{u'}.$$

Consequently,

$$\begin{aligned} q &= -v + u'\delta, & p &= u + v'\delta, \\ \xi &= -u + v'\delta, & \eta &= v + u'\delta. \end{aligned} \quad (12)$$

Next, let \mathfrak{M} denote the set of pairs of integers (v, u) on the segment $[2, 2\sqrt{x}]$ for which the numbers

$$0, \quad 2v, \quad v - u + \delta(v' - u'), \quad v + u + \delta(v' - u')$$

are distinct from one another and divisible by D .

Then, in view of (5), (11), and (12), the quantity $Q'(x)$ does not exceed the number of primes

$$q \equiv l \pmod{D}, \quad q \leq \sqrt{x},$$

for which the numbers

$$q + 2v, \quad q + v - u + \delta(v' - u'), \quad q + v + u + \delta(v' - u')$$

are also prime, as u and v independently run through pairs (v, u) belonging to the set \mathfrak{M} .

Thus, applying the lemma with $z = \sqrt{x}$, $m = 3$, and observing that the product over primes on the right-hand side of inequality (α) is a quantity of order L^ε , where $\varepsilon > 0$ is arbitrarily small, we obtain

$$\begin{aligned} Q'(x) &= B \frac{\sqrt{x}}{DL^{4-\varepsilon}} \sum_{(v,u) \in \mathfrak{M}} 1 = B \frac{\sqrt{x}}{DL^{4-\varepsilon}} \sum_{\substack{Dk \leq \sqrt{x} \\ Dk = v - u + \delta(v' - u')}} \sum 1 \\ &= \frac{B\sqrt{x}}{DL^{4-\varepsilon}} \sum_{1 \leq k \leq \sqrt{x}D^{-1}} \tau(Dk), \end{aligned}$$

where $\tau(n)$ is the number of positive divisors of n .

Further, we have

$$\sum_{n \leq z} \tau(n) \ll z \ln z, \quad \tau(Dk) \leq \tau(D)\tau(k), \quad \tau(D) \ll L^\varepsilon.$$

Consequently,

$$Q'(x) \ll xD^{-2}L^{-3+\varepsilon}. \quad (13)$$

Asymptotics of $Q(x)$. Applying the prime number theorem in arithmetic progressions, we find that

$$\begin{aligned} Q(x) &= \frac{1}{2} \sum_{p^2+q^2 \leq x} 1 + B\sqrt{x} = \frac{1}{\varphi(D)L} \sum_{p \leq \sqrt{x-4}} \sqrt{x-p^2} + \\ &+ B \frac{1}{\varphi(D)L^2\sqrt{x}} \sum_{p^2 \leq x} p^2 + B \frac{\sqrt{x}}{\varphi(D)} \sum_{2 \leq n \leq \pi(\sqrt{x}, D, l)} \frac{1}{\ln^2 n} + B\sqrt{x}. \quad (14) \end{aligned}$$

On the basis of Abel' s transformation, the first sum on the right-hand side of (14) is written in the form

$$\frac{2}{L} \sum_{1 \leq n \leq \sqrt{x}} (x - n^2)^{1/2} \{\vartheta(n, D, l) - \vartheta(n - 1, D, l)\} + B \frac{x}{\varphi(D)L^2}, \quad (15)$$

and the next two are estimated as

$$Bx(\varphi^2(D)L^3)^{-1}. \quad (16)$$

Next, we express the sum $S(x, D)$ from (15) in the form of the integral (4) and, using the known formula for Chebyshev's function $\vartheta(n, D, l)$, obtain

$$S(x, D) = \frac{x}{2\varphi(D)} \int_0^1 (1-z)^{1/2} z^{-1/2} dz + B \frac{x}{\varphi(D)L}. \quad (17)$$

But the integral in (17) is the B -function at the point (1.5; 0.5), so that combining estimates (10), (13)–(17) gives Theorems 1–3.

Theorem 4 is derived from Theorem 1 of the paper [7].

4. The method set forth makes it possible, at the cost of some complications in the calculations, to study also the question of the distribution of prime points (p, q) on ellipses $ap^2 + bq^2 = n$.

Theorem 5. The number $E(x, D)$ of numbers $n \leq x$ representable in the form

$$n = ap^2 + bq^2, \quad ()$$

with primes p and q from the progression $Dk + l$, where $(l, D) = 1$, $l < D \leq (\ln x)^M$, and with integers $a > b \geq 1$, is expressed by the formula

$$E(x, D) = \frac{\pi}{\varphi^2(D)\sqrt{ab}} \frac{x}{\ln^2 x} \left(1 + O\left(\frac{1}{\ln^{1-\varepsilon} x}\right) \right),$$

and $R(x, D)$, the number of numbers $n \leq x$ for which equation (γ) has one and only one solution, satisfies the relation

$$R(x, D) = E(x, D) + O\left(\frac{x}{D^2 \ln^{3-\varepsilon} x}\right).$$

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