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Abstract

Full Text

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ON CHEBYSHEV SETS IN BANACH SPACES

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We use the terminology introduced by N. V. Efimov and S. B. Stechkin ([1,2]).

A set M of a Banach space X is called **Chebyshev** if for every point $x \in X$ there is in M a unique point y nearest to x, i.e., such that $\|x - y\| = \rho(x, M)$. The point y is then called the **projection** of the element x onto the set M. A Chebyshev set M is called a **sun** if, for every point $x \in X$, the entire ray yx (issuing from y and passing through x) is projected into the point y—the projection of x onto M. A set is called **boundedly compact** if its intersection with every closed ball is compact in itself.

Under certain conditions the problem of Chebyshev sets admits an extremely simple solution, if one uses the fixed-point principle. (For an application of this principle to Chebyshev sets, see also the paper of V. Klee ([3]).)

Theorem. *In an arbitrary Banach space every boundedly compact Chebyshev set is a sun.*

An analogous proposition was proved by N. V. Efimov and S. B. Stechkin in the case of a uniformly convex space.

Proof. Suppose the contrary, i.e., that the set M is not a sun. Then there exists a point x such that the ray yx (y is the projection of x onto M) is not entirely projected into the point y. Obviously, on the ray yx there is a point farthest from y that is still projected into y (for convenience denote it again by x). Consider some closed ball V with center at the point x, not intersecting the set M. Denote its bounding sphere by S.

Introduce a mapping φ $(z \to z'')$ of the ball V into itself as follows. For a point $z \in V$ find its projection z'; then, of the two points of intersection of the ray $z'x$ with the sphere S, take the point z'' lying outside the segment $[z', x]$, and put $\varphi(z) = z''$. It is clear that φ is a continuous mapping (otherwise, for a point of discontinuity, by virtue of the bounded compactness of the set M, there would exist two nearest points in M) and that the set $\varphi(V)$ is compact (since M is boundedly compact).

By Schauder' s principle, in a Banach space every continuous mapping of a convex closed set into its compact part has a fixed point (see, for example, ([4]), p. 578). In our case the closed ball V is mapped into its compact part, $\varphi(V)$. Therefore there exists a point z_0 such that $z_0 = \varphi(z_0)$. If z_0' is the projection of z_0 onto M, then this equality means that the segment $[z_0, z_0']$ contains the point

x and $z_0 \in S$. But then the point z_0' will also be nearest to x, and since y is also nearest to x, it follows that $z_0' = y$. The point z_0 lies on the ray yx outside the segment $[y, x]$ and is projected into y. This is impossible by the assumption that x is the last point of the ray yx projected into y. Thus the theorem is proved.

Lemma. *In a smooth Banach space every sun is a convex set.*

This fact was noted by N. V. Efimov and S. B. Stechkin. For completeness we give the proof.

Suppose that the sun M is not convex. Then there exist points a, b, c such that

$$a \in M, \qquad b \in M, \qquad c \notin M, \qquad c \in [a, b].$$

Let the point nearest to c in M be the point p. At least one of the segments $[a, p]$, $[b, p]$ intersects the interior of the ball V with center at c and radius $\|c - p\|$. Otherwise, on the basis of the Hahn–Banach theorem, through each of the segments $[a, p]$, $[b, p]$ one could draw a supporting hyperplane to the ball V. At the point p of the ball we would then have two supporting hyperplanes, and this would mean nonsmoothness of the space, contrary to the assumption. Thus, let, for example, the segment $[b, p]$ intersect the interior of the ball V. Then for some ball V', similar to the ball V with center of similarity at the point p, the point b is an interior point. But the center c' of the ball V' lies on the ray pc, and the point b is closer to c' than p is. This contradicts the fact that M is a sun. The lemma is proved.

From the facts proved there follows the following

Theorem. *In every smooth Banach space every Chebyshev boundedly compact set is convex.*

Special cases of this theorem were proved in ([2,3]).

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Note: Figure translations are in progress. See original paper for figures.