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Abstract

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p-CONDUCTIVITY AND EMBEDDING THEOREMS FOR CERTAIN FUNCTIONAL SPACES INTO THE SPACE C

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MATHEMATICS

1°. In the note ([1]) sufficient, and in some cases necessary and sufficient, conditions were formulated for a domain of n-dimensional space E_n under which various embedding theorems hold. In the present work new theorems in this direction are given*.

Let D be an open set of the space E_n. The boundary of an arbitrary set E will be denoted by ΓE.

Definition 1. By the space $L_p^{(1)}(D)$ $(\widetilde{L}_p^{(1)}(D))$ $(p \geq 1)$ we shall mean the closure of the space $C^{(1)}(D)$ $(C^{(1)}(\overline{D}))$ of functions continuously differentiable in $D, \overline{D}$, with respect to the norm

$$\|u\|_{L_p^{(1)}(D)} = \|\operatorname{grad} u\|_{L_p(D)}.$$

Definition 2. Let E and F be subsets of D such that F is an open set, E is a set closed in D**, and $E \subset F$. The set

$$K = F \setminus E$$

will be called a **conductor** in D.

Definition 3. Let K be an arbitrary conductor in D. We shall say that a function $u(x)$ **belongs to the set** $U(K)$ if: 1) $u(x)$ is continuously differentiable in K; 2) $u(x) = 1$ on the set $\Gamma E \cap D$ and $u(x) = 0$ on the set $\Gamma F \cap D$; 3) $0 \leq u(x) \leq 1$ inside K.

The closure of $U(K)$ in the norm $L_p^{(1)}(K)$ $(p > 1)$ will be denoted by $U_p(K)$. From the well-known theorem of Whitney ([2]) on the possibility of extending a sufficiently smooth function given on an arbitrary closed set to the whole space with preservation of smoothness, it follows easily that the set $U(K)$ is nonempty.

Definition 4. Let K be an arbitrary conductor. The n-dimensional p-conductivity of the conductor K $(p > 1)$ is the number

$$c_p^{(n)}(K) = a_{p,n} \inf_{u(x) \in U_p(K)} \int_K |\operatorname{grad} u|^p \, dx,$$

where

$$a_{p,n} = \omega_n^{-1} \left(\frac{p-1}{|\, n-p \,|} \right)^{p-1} \quad \text{for } n \neq p, \qquad a_{n,n} = \omega_n^{-1};$$

ω_n is the surface area of the unit sphere in the space E_n.

Definition 5. If the set $\Gamma K \cap D$ (see Definition 2) is closed, then we shall sometimes call K a **condenser**. Instead of the term p-conductivity, in this case we shall speak of the p-capacity of the condenser. If the sets D and F coincide with the entire space E_n, then, naturally, one uses the term p-capacity of the set $E * * *$.

In proving the theorems of the work ([1]) and of the present note, use is made of the direct consequence, formulated below, of the well-known formula

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$*$ The definitions and notation of the note ([1]) are somewhat changed here.

$**$ A subset E of a set D is called closed in D if all points of D that are limit points for E belong to E.

$* * *$ On 2-conductivity and 2-capacity see ([3]); the definitions in ([3]) differ somewhat from those given above.

A. S. Kronrod ([4]) for the n-dimensional variation of a continuously differentiable function.

Lemma 1. Let $u(x)$ be a function defined and continuously differentiable in D, and let $\Phi(x)$ be a measurable function in D. Then

$$\int_{E_c} \Phi(x) |\operatorname{grad} u(x)|\, dx = \int_c^{+\infty} dt \int_{S_t} \Phi(x)\, d\sigma(x),$$

where $E_c = \{x;\ u(x) \geq c\}$; $S_t = \{x;\ u(x) = t\}$; $\int_{S_t} \cdots d\sigma(x)$ is the integral over the set S_t with respect to $(n-1)$-dimensional Hausdorff measure.

We give some properties of p-conductivity.

1. The n-dimensional p-conductivity of a conductor K can be defined by the equality

$$c_p^{(n)}(K) = a_{p,n} \inf_{u(x) \in U(K)} \left\{ \int_0^1 \frac{dt}{\left(\int_{S_t} |\operatorname{grad} u|^{p-1}\, d\sigma \right)^{\frac{1}{p-1}}} \right\}^{1-p}.$$

2. For the p-capacity of a condenser K the estimate holds

$$c_p^{(n)}(K) \geq \begin{cases} v_n^{\frac{p-n}{n}} \left| \mathrm{mes}_n^{-\frac{p-n}{n(p-1)}}(F) - \mathrm{mes}_n^{-\frac{p-n}{n(p-1)}}(E) \right|^{1-p}, & \text{if } n \neq p, \\ \left(\ln \dfrac{\mathrm{mes}_n F}{\mathrm{mes}_n E} \right)^{1-n}, & \text{if } n = p, \end{cases}$$

where v_n is the volume of the n-dimensional ball of unit radius. The equality sign is attained if the condenser is bounded by two concentric spheres. In particular, the capacity of an n-dimensional ball of radius R is equal to R^{n-p} for $n > p$ and to 0 for $n \leq p$.

3. Let $c_p^{(n)}(K) < \infty$. There exists a unique element $u^*(x) \in U_p(K)$ such that

$$c_p^{(n)}(K) = a_{p,n} \int_K |\operatorname{grad} u^*|^p dx.$$

For the element $u^*(x)$, for almost all t,

$$\int_{S_t} |\operatorname{grad} u^*|^{p-1} d\sigma = \text{const}.$$

Definition 6. Consider two conductors: $K = F \setminus E$, $K' = F' \setminus E'$. We shall say that the conductor K' is a **part** of K ($K' \subset K$), if $E \subseteq E' \subset F' \subseteq F$.

We formulate two more properties of p-conductivity:

4. If $K' \subset K$, then $c_p^{(n)}(K) \leq c_p^{(n)}(K')$.
5. For every $\varepsilon > 0$, for an arbitrary conductor K having finite p-conductivity, one can construct a conductor $K' \subset K$ such that

$$c_p^{(n)}(K') \leq c_p^{(n)}(K) + \varepsilon.$$

2°. The concept of p-conductivity turns out to be very useful in studying the connection between embedding theorems and sets on which functions are defined.

Definition 7. A set D belongs to the class $I_p^{(n)}$ ($p > n$), if there exists a positive constant $M < \mathrm{mes}_n D$ such that

$$\inf c_p^{(n)}(K) = \mathfrak{P}(M) > 0,$$

where the infimum is taken over all conductors K in D such that $\mathrm{mes}_n F \leq M$.

It can be shown that Definition 7 is equivalent to the following definition:

Definition 7 . A set D belongs to the class $I_p^{(n)}$ if there exists a positive constant R such that

$$\inf c_p^{(n)}(K_R) = \mathfrak{P}'(R) > 0.$$

Here the infimum is taken over all conductors $K_R = F \setminus E$ in D, where $F = S_R \cap D$, and E is the center of the n-dimensional ball S_R of radius R.

In note ([1]) a sufficient condition was formulated for the boundedness and complete continuity of the embedding operator $\widetilde{L}_p^{(1)}(D) \cap L(D)$ into $C(\overline{D})$. In terms of the classes $I_p^{(n)}$ one can formulate a necessary and sufficient condition. Namely, the following theorem is true:

Theorem 1. *If a set D belongs to the class $I_p^{(n)}$ and there exist constants $\mathfrak{R}(M) > 0$ and $k > 0$ such that for all $M' \leq M$*

$$\mathfrak{P}(M')\,(M')^k \geq \mathfrak{R}(M), \tag{1}$$

then almost everywhere in D the inequality

$$|u| \leq \{k_1 \| \operatorname{grad} u\|_{L_p(D)} + k_2 \|u\|_{L(D)}\}^{\frac{p}{k+p}} \|u\|_{L(D)}^{\frac{k}{k+p}} \tag{2}$$

holds.

The functions $u(x) \in L_p^{(1)}(D) \cap L(D)$ $[u(x) \in \widetilde{L}_p^{(1)}(D) \cap L(D)]$ coincide almost everywhere with functions continuous in D $[\overline{D}]$.

Conversely, if for every function $u(x) \in L_p^{(1)}(D) \cap L(D)$ inequality (2) holds, then the set D belongs to the class $I_p^{(n)}$ and (1) holds.

Corollary 1. *For boundedness of the embedding operator $L_p^{(1)}(D) \cap L(D)$ into $C(D)$, where $C(D)$ is the space of functions continuous and uniformly bounded in D, it is necessary and sufficient that $D \in I_p^{(n)}$.*

One can give an example of a domain D ($\operatorname{mes}_n D < \infty$) such that the embedding operator $L_p^{(1)}(D) \cap L(D)$ into $C(D)$ is bounded, but not completely continuous.

Below we formulate a criterion for complete continuity of the embedding operator $\widetilde{L}_p^{(1)}(D) \cap L(D)$ into $C(\overline{D})$.

Definition 8. A set D belongs to the class $\overline{I}_p^{(n)}$ $(p > n)$ if $D \in I_p^{(n)}$ and $\mathfrak{P}(M) \to \infty$ as $M \to 0$.

One can define the class $\overline{I}_p^{(n)}$ by the condition $\mathfrak{P}'(R) \to \infty$ as $R \to 0$.

Theorem 2. *Let $\operatorname{mes}_n D < \infty$. For complete continuity of the embedding operator $\widetilde{L}_p^{(1)}(D) \cap L(D)$ into $C(\overline{D})$ it is necessary and sufficient that $D \in \overline{I}_p^{(n)}$. If $D \in I_p^{(n)}$, then the embedding operator $L_p^{(1)}(D) \cap L(D)$ into $C(D)$ is completely continuous.*

Remark 1. It is easy to show that if at each point P of the set D one can construct a cone lying inside D and given, for a suitable choice of a rectangular coordinate system with origin at the point P, by the inequalities

$$\left\{ \left(\sum_{i=1}^{n-1} x_i^2 \right)^{1/2} < \varepsilon x_n^\beta,\ 0 < x_n < \rho \right\} \quad (\beta \geq 1),$$

where ε, ρ, β are fixed, then $D \in I_p^{(n)}$, and (1) holds for

$$k = -1 + \frac{p}{\beta(n-1)+1}.$$

Definition 9. A set D belongs to the class $J_{\nu(t)}^{(n)}$ if there exists a positive constant $M < \operatorname{mes}_n D$ and a function $\nu(t) \geq 0$ such that, for almost all t for which $\operatorname{mes}_n E_t \leq M$,

$$\sup \frac{\nu(\operatorname{mes}_n E_t)}{\operatorname{mes}_{n-1} S_t} = \mathfrak{B}(M) < \infty,$$

where the sup is taken over all functions $u(x) \in C^{(1)}(D) \cap L_1^{(1)}(D)$. For $\nu(t) = t^\alpha$ $\left(\alpha \geq \frac{n-1}{n}\right)$, following remark ([1]), we shall denote the class $J_{\nu(t)}^{(n)}$ by $J_\alpha^{(n)}$.

Let us give an example of a domain in the class $J_{\nu(t)}^{(n)}$.

Example 1. The n-dimensional cone

$$\left\{ \left(\sum_{i=1}^{n-1} x_i^2 \right)^{1/2} < f(x_n),\ 0 < x_n < 1 \right\},$$

where $f(x) \in C^{(1)}([0,1])$, $f'(x)$ is an increasing function satisfying the condition $f(0) = 0$, belongs to the class $J_{\nu(t)}^{(n)}$ if the function $\nu(t)$ satisfies the condition

$$\nu\left(\alpha_{n-1} \int_0^x f^{n-1}(\tau)\, d\tau \right) \leq f^{n-1}(x).$$

Here α_{n-1} is the $(n-1)$-dimensional Lebesgue measure of the $(n-1)$-dimensional ball of unit radius.

Theorem 3. Let $D \in J_{\nu(t)}^{(n)}$ and let $\Phi(t)$ be an N-function ([5]) such that

$$\int_0^M \Psi\left(\frac{1}{\nu(t)} \right) dt < \infty,$$

where $\Psi(t)$ is the function complementary to $\Phi(t)$, and let $u(x) \in C^{(1)}(D) \cap L(D)$. Then from the boundedness of the integral

$$\int_D \Phi(|\operatorname{grad} u|)\, dx \tag{3}$$

it follows that the function $u(x)$ is bounded. If $\text{mes}_n D < \infty$, then the set of functions for which the integral (3) is bounded is compact in $C(D)$.

Corollary 2. Let $D \in J_\alpha^{(n)}$ $(\alpha < 1)$ and

$$\int_D |\operatorname{grad} u|^{\frac{1}{1-\alpha}} \left(\ln^+_{m+1} |\operatorname{grad} u|\right)^r \left(\prod_{i=-1}^{m} \ln^+_i |\operatorname{grad} u|\right)^{\frac{\alpha}{1-\alpha}} dx < \infty, \qquad (4)$$

where $m \geq -1$ is an integer; $r > \frac{\alpha}{1-\alpha}$; $\ln^+_i t = \ln^+(\ln^+_{i-1} t)$ for $i \geq 2$; $\ln^+_1 t = \ln t$ for $t \geq 1$; $\ln^+_1 t = 0$ for $t < 1$; $\ln^+_0 t = \ln^+_{-1} t = 1$. Then the function $u(x)$ is bounded in D. If $\text{mes}_n D < \infty$, then the set of functions for which the integral (4) and $\|u\|_{L(D)}$ are bounded is compact in $C(D)$.

For $r = \frac{\alpha}{1-\alpha}$ Corollary 2 is false. Let us also note that if $D \in J_1^{(n)}$, then the embedding operator $L_p^{(1)}(D) \cap L(D)$ into $C(D)$ can be unbounded for all p.

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