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1961

SovietRxiv

Abstract

Full Text

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ON NEW EXTREMAL PROBLEMS

(Presented by Academician S. L. Sobolev on 26 IX 1960)

§ 1.

Modern mathematical cartography raises the question of constructing projections with a prescribed distribution of distortions in them ([1]). One of the basic requirements imposed on the projections sought is the minimality of the distortions within the territory being represented. The latter may be understood in various ways; however, all the different measures of the advantageousness of projections from this point of view can be combined into two groups: 1) criteria of variational type; 2) criteria of minimax type.

The search for projections satisfying criteria of the first type can be carried out on the basis of the well-known rules of the calculus of variations, whereas the construction of such projections on the basis of criteria of minimax type leads to new mathematical problems.

A criterion of minimax type was first formulated by P. L. Chebyshev ([2]) as applied to conformal projections; the theorem he stated without proof concerning the corresponding boundary conditions was proved by D. A. Grave ([3]). Some cartographers ([4,5]) put forward, with respect to equivalent projections, proposals analogous to Chebyshev' s theorem, in which, however, no attention was paid to the correctness of the formulation of the corresponding problem.

The formulation of new extremal problems of cartography has applied and scientific significance, which is why they are formulated here. At present the solution of one particular case of one of such problems is known ([2,3]). Below we give a particular case of another such problem, connected with the construction of the best Euler projection.

§ 2.

With respect to the quasilinear system, hyperbolic in the domain $D : |x| \leq A,\ |y| \leq \pi,\ A = 1 - \varepsilon$ ($\varepsilon > 0$ an arbitrarily small number),

$$e^{2v}(1-x^2)u_x + v_y = 0, \qquad u_y + e^{2v}(1-x^2)v_x = xe^{2v} \tag{1}$$

the following problem is posed. What initial values of the unknown functions

$$u = u(x,0) = \varphi(x), \qquad v = v(x,0) = \psi(x), \tag{2}$$

must be prescribed so that the solution of system (1), $v = v(x,y)$, deviates least from zero in some neighborhood of the initial curve $y = 0$?

We shall solve it on the basis of the method of characteristics. Along the characteristics C_1 and C_2 of system (1), determined by the equations

$$\frac{dy}{dx} = \pm\frac{1}{(1-x^2)e^{2v}}, \tag{3}$$

the relations

$$v + u + \frac{1}{2}\ln(1-x^2) = \text{const} = \xi, \qquad v - u + \frac{1}{2}\ln(1-x^2) = \text{const} = \eta. \tag{4}$$

hold.

If, for certain **given** initial values of the unknown functions for (1), it has been possible to construct a net of characteristics, then the values of the functions u and v at an arbitrary point P (Fig. 1) can be expressed in terms of their values at the points of the initial line P_1 and P_2; namely, from (4) we have

$$v_P = \frac{1}{2}(\xi_P + \eta_P) - \frac{1}{2}\ln(1-x_P^2), \tag{5}$$

where

$$\xi_P = \xi_{P_1} = v_1 + u_1 + \frac{1}{2}\ln(1-x_1^2), \qquad \eta_P = \eta_{P_2} = v_2 - u_2 + \frac{1}{2}\ln(1-x_2^2). \tag{6}$$

From (5) and (6) there follows the estimate

$$|v_P| < \frac{1}{2}|v_1| + \frac{1}{2}|v_2| + \frac{1}{2}|u_1| + \frac{1}{2}|u_2| +$$

$$+\frac{1}{4}|\ln(1-x_1^2)| + \frac{1}{4}|\ln(1-x_2^2)| + \frac{1}{2}|\ln(1-x_P^2)|. \tag{7}$$

If the point P is fixed, then, for **given** initial values, u and v, x_1 and x_2 have definite values which depend continuously on the initial data*; therefore, weakening the estimate (7) somewhat, we introduce into it $K = |\ln(1-A^2)| = \text{const}$. Then for any point lying in the domain of influence of the interval $[-A, +A]$, under the prescribed initial conditions,

Fig. 1

Figure 1: Fig. 1

$$2|v_P| < |v_1| + |v_2| + |u_1| + |u_2| +$$

$$+K + |\ln(1 - x_P^2)|. \tag{8}$$

Fig. 1

The estimate (8) also holds at that point M at which $|v_M| = \max$. Wishing to find functions (2) leading to $\min\max|v|$, one must, by varying the initial data, strengthen the estimate (8).

The most effective strengthening of the inequality (8) can be obtained by putting $v_1 = v_2 = u_1 = u_2 = 0$, and since the positions of the points P_1 and P_2 change when the initial data are changed, we are forced finally to accept

$$u = \varphi(x) \equiv 0, \qquad v = \psi(x) \equiv 0. \tag{9}$$

The solution of the given problem also leads to the conclusion that, under the conditions (9), the solution of system (1) at **any** point P of a neighborhood of $y = 0$ has the smallest value in absolute magnitude among all those values which it can take at this point for **any other** Cauchy initial data. The result obtained is not only local in character relative to a neighborhood of the straight line $y = 0$, but also global–for the entire domain of influence of the interval $[-A, +A]$ under (9).

Solving the problem for the corresponding system (1) of a quasilinear partial differential equation of the second order, we would likewise obtain zero initial conditions, i.e. the sought integral surface would be tangent to the plane xOy along the initial line $y = 0$. In solving analogous problems for quasilinear hyperbolic equations of the second order, this fact will hold in all those cases when, on the basis of some considerations, it can be established that all integral surfaces of the given equation consist entirely of points of elliptic type.

* A change of the initial conditions leads to a change in the domain of influence of the interval $[-A, +A]$, which we do not take into account, since the problem is solved for a neighborhood of this interval. If system (1) were linear, the indicated circumstance would not occur.

§ 3. Let us formulate new extremal problems.

A. Suppose we have the quasilinear system

$$a_{11}u_x + a_{12}v_x + b_{11}u_y + b_{12}v_y = c_1, \qquad a_{21}u_x + a_{22}v_x + b_{21}u_y + b_{22}v_y = c_2, \text{ (I)}$$

whose coefficients a_{ij}, b_{ij} and free terms c_i are assumed to be continuous functions of x, y, u, v in some domain D.

If system (I) is elliptic in any partial domain $\Delta \in D$, then we have the following generalization of Chebyshev' s problem: what boundary conditions $u = f_1(s)$ and $v = f_2(s)$, where s is the arc parameter, must be prescribed so that one of the sought functions (u or v) deviates least from zero in Δ?

Suppose system (I) is hyperbolic in some domain D. It is required to determine what initial values $u = f_1(s)$ and $v = f_2(s)$ must be prescribed on a given nowhere characteristic curve L, so that in the domain $\Delta \in D$ one of the functions u or v deviates least from zero. Here only such variations of the initial functions f_1 and f_2 are allowed that the domain of their influence entirely covers the domain Δ.

If system (I) belongs in the domain D to the parabolic type, or even to some mixed type, then analogous problems are posed: what additional conditions must be prescribed so that one of the sought functions deviates least from zero in the domain Δ, partial with respect to the domain D.

In the formulated problems, continuity conditions are imposed on the functions f_1 and f_2 and on their first partial derivatives.

B. We have presented problems with respect to system (I), but, of course, problems on determining the form of additional conditions under which the sought function deviates least from zero in a given domain can also be posed for quasi-linear equations of the 2nd order and, in general, for equations (and systems) of various orders of general form.

C. The indicated groups of problems can be generalized in the following way. We shall assume that it is necessary to find not only the additional conditions, but also the curve (or curves) on which they must be prescribed, so that the sought solution satisfies the minimax requirement. Natural restrictions must be imposed on these curves, concerning their smoothness, noncharacteristic nature, etc., as determined by the detailed formulation of the problem and by the type of the equation. Such problems are especially difficult for equations of elliptic type, for in this case it is necessary to find also a closed curve in Δ and the values of the sought function on it, and then to solve simultaneously the interior and exterior problems with respect to this curve, whose solutions must be identical on this curve. The latter problems are at present posed on the basis of entirely different considerations ([6]).

§ 4. In all the problems presented, the additional conditions and the curves bearing them must be found under the condition that the function satisfying the given differential equation (or system of such equations), for example the function v, deviates least from zero in the given domain of variation of the independent variables Δ, which requires clarification.

Let us denote, for some equation (or system of equations), by S the aggregate of additional conditions that provides a correct formulation of the problem for the given equation in some domain of variation of the independent variables Δ. From the infinite set of all conceivable S_i, evidently, one can always choose such an S_0 which, from all particular solutions of the equation, singles out one satisfying some preassigned condition. In our problems such a condition is the fulfillment of the minimax principle, which may be understood in two ways.

Suppose that, from the whole set of functions continuous together with all their partial derivatives up to order n inclusive in some

of a given domain Δ, a subset K is singled out, each element v_i of which is a solution of a given partial differential equation of order n.

In the classical sense, the function $v_0 \in K$ that deviates least from zero in the domain Δ is the one for which $\max |v_0|_{P_0} = \min$, where $P_0 \in \Delta$ is some point; at other points $P \in \Delta$ the possibility is not excluded that $|v_i|_P < |v_0|_P$.

For some classes of partial differential equations, the notion of their solutions deviating least from zero can be strengthened. Namely, let us call a solution $\tilde{v}_0$ of the equation under consideration in the domain Δ a function deviating least from zero in this domain if at all its points $|\tilde{v}_0| < |v_i|$, where v_i are any other solutions of the equation distinct from $\tilde{v}_0$. Examples of such functions (up to an additive constant) may be, in a neighborhood of the initial curve on which zero Cauchy data are prescribed, solutions of such equations all of whose integral surfaces everywhere have total curvature $\chi > 0$. It is clear that a solution of an equation satisfying the strengthened minimax principle automatically satisfies it in the classical formulation.

§ 5. The existence of solutions of the indicated problems follows directly from the definition of well-posedness of the formulation of those problems of mathematical physics with respect to which the new extremal problems are posed; but the conditions under which the given equation (or system of such equations) admits a solution satisfying the strengthened minimax principle must be proved. The same also applies to questions of uniqueness of solutions of the posed problems.

§ 6. The considerations set forth above lead to great possibilities for extending the theory of partial differential equations. Indeed, if in mathematical physics until now various methods for solving problems with given additional conditions were mainly considered, then now problems arise concerning the preliminary establishment of additional conditions under which the sought solution of the equation satisfies some definite requirement (minimax or some other). In other words, if earlier the theory of differential equations was a theory of physics (mathematical physics), a theory of the description of physical processes and phenomena, then now (after practice has posed new problems, at least of the type indicated here) the theory of differential equations becomes a theory of prediction and control of the processes and phenomena described by these equations.

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Received
24 IX 1960

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Note: Figure translations are in progress. See original paper for figures.