



Soviet-era science, translated into English

MATHEMATICS

D. L. BERMAN

1961

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196101.78285>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

MATHEMATICS

D. L. BERMAN

EXTREMAL PROBLEMS IN THE THEORY OF POLYNOMIAL OPERATORS

(Presented by Academician A. N. Kolmogorov, 21 I 1961)

1. The present note is devoted to extremal problems in the theory of polynomial operators for the space \tilde{C} of 2π -periodic continuous functions with norm $\|f\| = \max_{0 \leq x < 2\pi} |f(x)|$. In what follows k is an integer, $k \geq 0$; n and m are natural numbers; $f^{(k)}(x)$ is the k -th derivative of $f(x)$; polynomials are everywhere understood to be trigonometric.
2. **Theorem 1.** Let $\Omega_n^{(k)}$ be the set of all linear operators $U_n(f, x)$ from \tilde{C} into \tilde{C} having the property

$$U_n(f, x) = f^{(k)}(x), \tag{1}$$

if $f(x)$ is a polynomial of order $\leq n$. Put $\rho_n^{(k)} = \inf_{U_n \in \Omega_n^{(k)}} \|U_n\|$. The equalities $\rho_n^{(k)} = n^k$, $k = 0, 1, 2, \dots$ are valid. For any k one can indicate such an operation $\bar{U}_n \in \Omega_n^{(k)}$ that $\|\bar{U}_n\| = \rho_n^{(k)}$.

Proof. For definiteness let us consider the case of even k . According to property (1) of the operators $U_n(f, x)$,

$$U_n(\cos n\theta) = \pm n^k \cos n\theta.$$

Therefore $\|U_n\| \geq n^k$. Hence $\rho_n^{(k)} \geq n^k$. Now consider the operator

$$U_n(f, x) = \sum_{j_1, j_2, \dots, j_k} f(x + x_{j_1} + x_{j_2} + \dots + x_{j_k}) r_{j_1} r_{j_2} \dots r_{j_k},$$

$$1 \leq j_s \leq 2n; \quad s = 1, 2, \dots, k, \tag{2}$$

where

$$r_{j_s} = (-1)^{j_s-1} \frac{1}{2n \sin^2 x_{j_s}/2}; \quad x_{j_s} = \frac{2j_s - 1}{2n} \pi.$$

With the aid of the well-known identity of Riesz (1) it is easy to verify that the operator (2) belongs to $\Omega_n^{(k)}$. Since (1)

$$\sum_{j_s=1}^{2n} |r_{j_s}| = n, \quad s = 1, 2, \dots, k,$$

the norm of the operator (2) is equal to n^k . Therefore, for $k \geq 1$, the operator (2) is extremal. If $k = 0$, then the identity operator is extremal.

Along with the set of operators $\Omega_n^{(k)}$, introduce the set $\Omega_{n,n}^{(k)}$ of linear operators $U_{n,n}(f, x)$ from \tilde{C} into \tilde{C} having the properties: 1) for any $f \in \tilde{C}$, $U_{n,n}(f, x)$ is a polynomial of order $\leq n$; 2) if $f(x)$ is a polynomial of order $\leq n$, then $U_{n,n}(f, x) = f^{(k)}(x)$.

Put

$$\rho_{n,n}^{(k)} = \inf_{U_{n,n} \in \Omega_{n,n}^{(k)}} \|U_{n,n}\|.$$

Since $\Omega_{n,n}^{(k)} \subset \Omega_n^{(k)}$, it follows that $\rho_{n,n}^{(k)} \geq \rho_n^{(k)}$. A natural question arises about the asymptotic estimate of the ratio $\rho_{n,n}^{(k)}/\rho_n^{(k)}$ as $n \rightarrow \infty$. The solution of this question is given by Theorem 2.

Theorem 2. *Let k be a fixed integer satisfying $k \geq 0$. Then the equality*

$$\lim_{n \rightarrow \infty} \left(\frac{\rho_{n,n}^{(k)}}{\rho_n^{(k)}} : \frac{4}{\pi^2} \ln n \right) = 1.$$

Let us now consider the set $\Omega_{n,n+m}^{(\Phi)}$ of linear operators $U_{n,n+m}(f, x)$ from \tilde{C} to \tilde{C} , possessing the following properties: 1) for every $f \in \tilde{C}$, $U_{n,n+m}(f, x)$ is a polynomial of order $\leq (n+m)$; 2) if T is a polynomial of order not higher than n , then

$$U_{n,n+m}(T, x) = \sigma^{(\Phi)}(T, x), \quad \sigma^{(\Phi)}(f, x) = \int_0^{2\pi} f(x+t)\Phi(t) dt, \quad (3)$$

where $\Phi(t)$ is a given polynomial of order $\leq (n+m)$.

If, as the polynomial $\Phi(t)$, one takes the Vallée-Poussin kernel $V_{n,m}(t)$,

$$V_{n,m}(t) = \frac{1}{\pi(m+1)} \sin \left(n + \frac{m+1}{2} \right) t \sin \frac{m+1}{2} t : \sin^2 \frac{t}{2},$$

then equality (3) takes the form

$$U_{n,n+m}(T, x) = T(x) \tag{4}$$

for every polynomial $T(x)$ of order not higher than n . The most important operation of the form $U_{n,n+m}(f, x)$ with property (4) is the Vallée-Poussin partial sum

$$\sigma_{n,m}^{(V)}(f) = \frac{1}{m+1} \sum_{k=n}^{n+m} S_k(f),$$

where $S_k(f)$ is the partial sum of order k of the Fourier series of the function $f(x)$. Among operations of the form $U_{n,n+m}(f, x)$ with property (4) are also the well-known interpolation processes of S. N. Bernstein (2).

Theorem 3. Let

$$\rho_{n,n+m}^{(\Phi)} = \inf_{U_{n,n+m} \in \Omega_{n,n+m}^{(\Phi)}} \|U_{n,n+m}\|.$$

Then the equality holds

$$\rho_{n,n+m}^{(\Phi)} = \inf_{\alpha_k, \beta_k} I_{\Phi}(\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_m),$$

where

$$\begin{aligned} I_{\Phi}(\alpha_1, \alpha_2, \dots, \alpha_m, \beta_1, \beta_2, \dots, \beta_m) = \\ = \int_0^{2\pi} \left| S_n(\Phi, t) + \sum_{j=1}^m (\alpha_j \cos(n+j)t + \beta_j \sin(n+j)t) \right| dt. \end{aligned} \tag{5}$$

If the integral (5) attains its minimum for $\alpha_j = \alpha_j^{(0)}$, $\beta_j = \beta_j^{(0)}$, $j = 1, 2, \dots, m$, then the extremal operation is computed by the formula

$$\bar{U}_{n,n+m}(f, x) = \int_0^{2\pi} f(x+t) \left[S_n(\Phi, t) + \sum_{j=1}^m (\alpha_j^{(0)} \cos(n+j)t + \beta_j^{(0)} \sin(n+j)t) \right] dt. \tag{6}$$

That is,

$$\|\bar{U}_{n,n+m}\| = \rho_{n,n+m}^{(\Phi)}.$$

We outline the proof of this theorem. In (3) it is proved that for every $U_{n,n+m} \in \Omega_{n,n+m}^{(\Phi)}$

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} U_{n,n+m}[f(z+t), x-t] dt &= \sigma^{(\Phi)}(S_n(f), x) + \\ &+ \frac{1}{2} \sum_{k=n+1}^{n+m} (A_k \cos kx + B_k \sin kx), \end{aligned} \quad (7)$$

where $A_k = \delta_k b_k + \gamma_k a_k$; $B_k = \delta_k b'_k - \gamma_k a_k$; a_k and b_k are the Fourier coefficients of $f(x)$; δ_k and γ_k are numbers depending only on $U_{n,n+m}$. Since $\sigma^{(\Phi)}(S_n(f), x) = S_n(\sigma^{(\Phi)}(f), x)$, equality (7) takes the form

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} U_{n,n+m}[f(z+t), x-t] dt &= \\ &= S_n(\sigma^{(\Phi)}(f), x) + \frac{1}{2} \sum_{k=n+1}^{n+m} (A_k \cos kx + B_k \sin kx). \end{aligned}$$

From this it is easy to derive that

$$\|U_{n,n+m}\| \geq \inf_{\alpha_k, \beta_k} I_{\Phi}(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m).$$

Thus,

$$\rho_{n,n+m}^{(\Phi)} \geq \inf_{\alpha_k, \beta_k} I_{\Phi}(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_m). \quad (8)$$

If the integral (5) attains its minimum for $\alpha_j = \alpha_j^{(0)}$, $\beta_j = \beta_j^{(0)}$, $j = 1, 2, \dots, m$, then for the operation $\bar{U}_{n,n+m}(f, x)$, defined by (6), we have

$$\|\bar{U}_{n,n+m}\| = I_{\Phi}(\alpha_1^{(0)}, \dots, \alpha_m^{(0)}, \beta_1^{(0)}, \dots, \beta_m^{(0)}). \quad (9)$$

From (8) and (9) theorem 3 follows.

Theorem 4. Let $\Omega_{n,n+m}^{(k)}$ be the set of operators $U_{n,n+m}(f, x)$ from \tilde{C} to \tilde{C} possessing the properties: 1) for every $f \in \tilde{C}$, $U_{n,n+m}(f, x)$ is a polynomial of order $\leq (n+m)$; 2) if $f(x)$ is a polynomial of order $\leq n$, then $U_{n,n+m}(f, x) = f^{(k)}(x)$. Put

$$\rho_{n,n+m}^{(k)} = \inf_{U_{n,n+m} \in \Omega_{n,n+m}^{(k)}} \|U_{n,n+m}\|.$$

Then

$$\rho_{n,n+m}^{(k)} = \inf_{\alpha_k, \beta_k} I(\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_k),$$

where*

$$\begin{aligned} I &= I(\alpha_1, \alpha_2, \dots, \alpha_k, \beta_1, \beta_2, \dots, \beta_k) = \\ &= \frac{1}{\pi} \int_0^{2\pi} \left| D_n^{(k)}(t) + \sum_{j=1}^m (\alpha_j \cos(n+j)t + \beta_j \sin(n+j)t) \right| dt. \end{aligned} \quad (10)$$

If the integral (10) attains its least value for $\alpha_j = \alpha_j^{(1)}$, $\beta_j = \beta_j^{(1)}$, $j = 1, 2, \dots, m$, then the extremal operation is

$$\begin{aligned} \bar{U} &= \bar{U}_{n,n+m}(f, x) = \\ &= \frac{1}{\pi} \int_0^{2\pi} f(x+t) \left[D_n^{(k)}(t) + \sum_{j=1}^m (\alpha_j^{(1)} \cos(n+j)t + \beta_j^{(1)} \sin(n+j)t) \right] dt. \end{aligned}$$

This theorem follows directly from theorem 3, if one observes that the set of operators $\Omega_{n,n+m}^{(k)}$ is a special case of the set of operators $\Omega_{n,n+m}^{(\Phi)}$, when $\Phi(t) = \frac{1}{\pi} D_n^{(k)}(t)$.

Of particular interest are the special cases of theorem 4 when $k = 0$ and $k = 1$. Let us first consider the case $k = 0$. Since $D_n(t)$ is an even function, $\rho_{n,n+m}^{(0)}$ is computed by the formula

$$\rho_{n,n+m}^{(0)} = \frac{2}{\pi} \inf_{\alpha} \int_0^{\pi} \left| D_n(t) + \sum_{j=1}^m \alpha_j \cos(n+j)t \right| dt. \quad (11)$$

* $D_n(t)$ is the Dirichlet kernel.

If the integral on the right-hand side of (11) attains its minimum for $\alpha_j = \tilde{\alpha}_j$, $j = 1, 2, \dots, m$, then the extremal operation is found from the formula

$$\bar{U} = \frac{1}{\pi} \int_0^{2\pi} f(x+t) \left[D_n(t) + \sum_{j=1}^m \alpha_j \cos(n+j)t \right] dt. \quad (12)$$

Thus, among all linear operations from \tilde{C} into \tilde{C} , taking functions from \tilde{C} into polynomials of order $(n + m)$ and preserving polynomials of order n , the operation (12) has the least norm.

As is known, the numbers $\tilde{\alpha}_j$ can be found from the equalities

$$\int_0^\pi \text{sign} \left[D_n(t) + \sum_{j=1}^m \tilde{\alpha}_j \cos(n + j)t \right] \cos(n + i)t dt = 0, \quad i = 1, 2, \dots, m.$$

From this, by direct verification, we see that, when n is a multiple of $(m + 1)$, the extremal operation coincides with the Vallée-Poussin partial sum $\sigma_{n,m}^{(V)}(f)$ (for this, see also (4)). Incidentally, this result follows directly from inequality (14) of note (3), if one takes into account the simple observation that, when n is a multiple of $(m + 1)$, the expansion of the function $\text{sign } V_{n,m}(t)$ in a Fourier series contains no terms with $\cos jt$, where $n + 1 \leq j \leq n + m$. Let us now consider the case when $k = 1$ and $m = n - 1$. With the help of Theorem 4 we obtain the theorem:

Theorem 5. Among all linear operators $U_{n,2n-1}(f, x)$ from \tilde{C} into \tilde{C} , taking functions from \tilde{C} into polynomials of order $(2n - 1)$ and possessing the property that for every polynomial of order $\leq n$ the equality $U_{n,2n-1}(f, x) = f'(x)$ holds, the operator

$$\bar{U}_{n,2n-1}(f, x) = \frac{1}{\pi} \int_0^{2\pi} f(x + t) \sin nt \left(\frac{\sin nt/2}{\sin t/2} \right)^2 dt \quad (13)$$

has the least norm. Thus:

$$\rho_{n,2n-1}^{(1)} = \|\bar{U}_{n,2n-1}\| = 2nF_{n-1}(|\sin nt|, 0),$$

where $F_{n-1}(f, x)$ is the Fejér mean of order $(n - 1)$.

As is known (5), the operator (13) was used by A. Zygmund to prove S. N. Bernstein's inequality for the modulus of the derivative of a trigonometric polynomial. As for the extremal property of this operator, expressed by Theorem 5, it apparently has not been noted until now. In the proof of the above-mentioned S. N. Bernstein inequality by means of the operator (13), the constant obtained was twice its exact value. From Theorem 5, in particular, it follows that among the operators $\Omega_{n,2n-1}^{(1)}$ there is no such operator by means of which one can prove S. N. Bernstein's inequality with the exact constant.

Remark. The question of extremality of the Vallée-Poussin partial sums is resolved by the following theorem:

For the Vallée-Poussin partial sum $\sigma_{n,m}^{(V)}(l)$ to have the least norm in the class of operators $\Omega_{n,n+m}^{(0)}$, it is necessary and sufficient that $2n$ be a multiple of $(m+1)$.

Received
14 I 1961

REFERENCES

1. V. L. Goncharov, *Theory of Interpolation and Approximation of Functions*, Moscow-Leningrad, 1934.
2. S. N. Bernstein, *Collected Works*, vol. 2, Publishing House of the Academy of Sciences of the USSR, 1954, p. 146.
3. L. Berman, DAN, 95, No. 2 (1954).
4. V. F. Nikolaev, DAN, 96, No. 1 (1954).
5. A. Zygmund, *Trigonometric Series*, 1939.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.