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## Abstract

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## MATHEMATICS

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# SEPARATRICES OF PLANE DYNAMICAL SYSTEMS

*(Presented by Academician P. S. Aleksandrov, 22 V 1961)*

1. Consider a dynamical system $f(p,t)$, defined by a system of differential equations ((3), Ch. I, § 3), whose right-hand sides are continuous and ensure uniqueness of the solution in the entire plane. We assume that the set of singular points of the system is isolated. For such dynamical systems Markus (1) introduced the concept of a separatrix and proved that the families of trajectories of two systems are $o$-homeomorphic if and only if their systems of separatrices are $o$-homeomorphic. In the present note the notion of a separatrix is reduced to the known notion of a trajectory that is orbitally unstable ((4), Ch. VI, § 3) and to a certain generalization of a trajectory continued in the sense of Bendixson (Theorems 7 and 7′).
2. Complete the plane $E^2$ by a point at infinity; the resulting space, homeomorphic to a sphere, will be denoted by $S^2$. The point $\infty$ is a singular point of the system $f(p,t)$—the only one which may fail to be an isolated singular point of this system.

Let $l_p^+$ denote the positive semitrajectory with initial point $p$, i.e.

$$l_p^+ = f(p;\, 0 \le t < +\infty).$$

The semitrajectories $l_p^+$ and $l_r^+$ are equivalent if the points $p$ and $r$ belong to one and the same trajectory. Denote by $\Lambda^+$ the set of all nonsingular (i.e. distinct from points) positive semitrajectories of the dynamical system $f(p,t)$. If $P \subset S^2$, then $l_P = \{l_p^+\}_{p\in P}$ is the family of all positive semitrajectories whose initial points belong to $P$. The set $l_P$ will be regarded as open if $\bigcup_{p\in P} l_p^+$ is an open set in $S^2$. By a neighborhood of a semitrajectory and by the limit of a sequence of semitrajectories we shall henceforth mean a neighborhood and a limit in the thus defined space $\Lambda^+$.

Analogous notation and definitions are introduced for negative semitrajectories. In what follows, as a rule, definitions and theorems are given only for positive semitrajectories; we omit, without mentioning it each time, the analogous formulations for negative semitrajectories.

3. A simple arc (i.e. the homeomorphic image of a closed interval) $\pi$ will be called a section if $\pi$ contains no singular points and intersects each trajectory of the system $f(p,t)$ in no more than one point. If $p_1, p_2$ are the endpoints of $\pi$, then put $\bar{\pi} = \pi \setminus \{p_1, p_2\}$. If $l$ is a trajectory and $l_p^+, l_r^-$ are its positive and negative semitrajectories, then $\omega(l) = \omega(l_p^+)$ denotes the $\omega$-limit set of $l$, and $\alpha(l) = \alpha(l_r^-)$ the $\alpha$-limit set of $l$. The geometric $\omega$-limit set of $l$ will be the set

$$\tilde{\omega}(l) = \tilde{\omega}(l_p^+) = \omega(l) \setminus l.$$

**Definition 1.** The semitrajectory $l_{p_0}^+$ is called **ordinary** if there exists a section $\pi$ such that $p_0 \in \bar{\pi}$ and such that: 1) if $p \in \pi$,

then $\tilde{\omega}(l_p^+) = \tilde{\omega}(l_{p_0}^+)$; 2) if $v = \bigcup_{p \in P} l_p^+$, then the boundary $v = \tilde{\omega}(l_p^+) \cup \bar{\pi} \cup l_{p_1}^+ \cup l_{p_2}^+$, where $p_1, p_2$ are the endpoints of the arc $\bar{\pi}$.

**Definition 2.** The semitrajectory $l_p^+$ is called a **semiseparatrix** if it is not an ordinary semitrajectory.

**Theorem 1.** *A trajectory $l$ is a separatrix in the sense of Markus if and only if at least one of the semitrajectories contained in $l$ is a semiseparatrix.*

**4. Definition 3** (4). The semitrajectory $l_{p_0}^+$ is called **orbitally stable** if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $p \in S(p_0, \delta)$, then $l_p^+ \subset S(l_{p_0}^+, \varepsilon)$. $S(R, \eta)$ denotes the set of points whose distance from $R$ is less than $\eta$.

If in Definition 3 distance is understood as Euclidean distance, then the positive semitrajectories of the system $\dot{x} = x$, $\dot{y} = y$ will all be orbitally unstable, although they possess no geometric singularities. Therefore it is natural to introduce on $S^2$ such a metric under which $S^2$ is isometric to the geometric sphere $x^2 + y^2 + z^2 = 1$ in $E^3$. A semitrajectory for which the conditions of Definition 3 are fulfilled will be called **orbitally stable in** $E^2$ or, respectively, **in** $S^2$, depending on which metric is considered: the Euclidean metric in $E^2$ or the above-defined metric in $S^2$. If $\infty \notin \omega(l_p^+)$, then $l_p^+$ is either orbitally stable both in $E^2$ and in $S^2$, or orbitally unstable both in $E^2$ and in $S^2$. A trajectory is called **orbitally unstable** if at least one of its semitrajectories is orbitally unstable. If $l$ is a closed trajectory, then $l$ is orbitally unstable if and only if it is a limit cycle (4). If $\omega(l_p^+) = q \neq \infty$, i.e. $l_p^+$ enters a finite singular point, then $l_p^+$ is orbitally unstable if and only if it is extendable in the sense of Bendixson with respect to some circle.

**Theorem 2.** *An ordinary trajectory is orbitally stable in $S^2$.*

**5.** From the definition of a separatrix it follows directly that a singular point is a separatrix. A closed trajectory is a separatrix if and only if it is a limit cycle (1).

**Definition 4.** The semitrajectory $l_p^+$ is a **semispiral** if $\tilde{\omega}(l_p^+) = \omega(l_p^+) \setminus l_p^+$ contains a nonsingular point. This name is suggested by the character of the approach of a semitrajectory of this type to its limit set ($^{5,6}$).

**Theorem 3.** *If $l_p^+$ is a semispiral and $\infty \notin \tilde{\omega}(l_p^+)$, then $l_p^+$ is an ordinary semitrajectory.*

**Theorem 4.** *A semispiral $l_p^+$ is a semiseparatrix only when $l_p^+$ is the limit of a sequence $\{l_{p_n}^+\}$ of semitrajectories extendable in the sense of Bendixson.*

**6. Definition 5.** Given a trajectory $l$, a point $q = \omega(l)$, a disk $K$ and its boundary $C$. Suppose: a) $q \in K$; b) $l \cap C \neq \emptyset$; c) $C$ contains no singular points. The semitrajectory $l_p^+ \subset l$ is called **extendable with respect to the circle** $C$ if there exists a trajectory $m$ such that: 1) $m \cap C \neq \emptyset$; 2) $\alpha(m) \subset K$; 3) if $m_{r_0} \subset m$; $l_{p_0}^+ \subset l$; $\overline{\pi}, \rho$ are sections; $p_0 \in \pi$, $r_0 \in \rho$, then there exists a sequence $p_n \in \pi$, $p_n \to p_0$ such that $l_{p_n}^+$ intersects $\rho$ at a point $r_n$ and the arcs of the trajectory $p_n r_n \subset K$. Any semitrajectory $m_r^- \subset m$ is then called an **extension** of $l_p^+$ with respect to $C$. The semitrajectory $l_p^+$ is called **extendable** if there exists a circle $C$ such that $l_p^+$ is extendable with respect to $C$.

**Theorem 5.** *Let $\omega(l) = q$ and suppose that for $K$ and $C$ the conditions a), b), c) of Definition 5 are fulfilled. Let $p_0 \in l$, $l_{p_0}^+ \subset K$, $p_0 p_1 = \overline{\pi}$ be a section. If* there exists a sequence $p_n \to p_0$, $p_n \in \pi$ such that $l_{p_0}^+ \cap C \neq \emptyset$, then $l_{p_0}^+$ is extendable with respect to $C$.

**Corollary 1.** If $\omega(l) = q$, then $l_p^+ \subset l$ is extendable if and only if it is orbitally unstable in $S^2$.

A nonextendable semitrajectory, even one entering an isolated singular point, may also be a separatrix.

**Theorem 6.** Let $\omega(l_p^+)$ be a singular point, and suppose that in some neighborhood of $l_p^+$ there are no extendable semitrajectories. Then $l_p^+$ is an ordinary semitrajectory.

**7. Theorem 7.** A semitrajectory $l_p^+$ is a semiseparatrix if and only if $l_p^+$ belongs to one of the following types: singular point, limit cycle, extendable semitrajectory, limit of limit cycles, limit of extendable semitrajectories.

**Theorem 7′.** A semitrajectory $l_p^+$ is a semiseparatrix if and only if either $l_p^+$ is a singular point, or the semitrajectory is orbitally unstable in $S^2$, or it is the limit of a sequence of semitrajectories orbitally unstable in $S^2$.

**8.** Let $U_m$ be the class of all dynamical systems defined by differential equations of the form

$$\dot{x} = X_m(x, y), \qquad \dot{y} = Y_m(x, y),$$

where $X_m, Y_m$ are polynomials of degree $m$ with no common linear factors. Theorem 7 makes it possible, on the basis of ([2],[7]), to draw the following conclusion from the Markus theorem mentioned at the beginning:

**Theorem 8.** The class $U_m$ contains only a finite number of nonhomeomorphic systems.

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*Note: Figure translations are in progress. See original paper for figures.*

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