MATHEMATICS

A. A. GOL'DBERG

1961

SovietRxiv

Abstract

Full Text

MATHEMATICS

A. A. GOL' DBERG

ON THE DISTRIBUTION OF VALUES OF MEROMORPHIC FUNCTIONS WITH SEPARATED ZEROS AND POLES

(Presented by Academician M. A. Lavrent' ev, 14 XI 1960)

Let $f(z)$ be a function meromorphic in $|z| < R \leqslant \infty$. We shall use the standard notation of Nevanlinna theory (^{1}). Denote

$$N(r) = N(r,\infty) + N(r,0), \qquad m(r) = m(r,\infty) + m(r,0).$$

All limits occurring below are taken as $r \to R$. In many investigations (see, for example, (^{2})) the quantity

$$k(f) = \limsup \frac{N(r)}{T(r)}, \qquad 0 \leqslant k(f) \leqslant 2,$$

is introduced to characterize the distribution of zeros and poles. The inequality $k(f) < 2$ indicates an "insufficient," in comparison with the "normal," number of zeros and poles of the function $f(z)$. It will be convenient for us to introduce the quantity

$$d(f) = 2 - k(f) = \liminf \frac{m(r)}{T(r)},$$

which it is natural to call the deficiency of zeros and poles of the function $f(z)$. Obviously, $0 \leqslant d(f) \leqslant 2$ and

$$d(f) \geqslant \delta(0) + \delta(\infty).$$

In the present note we shall show that certain restrictions imposed only on the arguments of the zeros and poles of $f(z)$ imply $d(f) > 0$. In addition, the question will be discussed of when, under these same restrictions,

$$\delta(0) + \delta(\infty) > 0.$$

Let p be a natural number, $\pi/(2p) > \eta \geqslant 0$, $0 \leqslant \varphi_0 < 2\pi$. Denote by $D_1^p(\eta,\varphi_0)$ and $D_2^p(\eta,\varphi_0)$ the following sets:

$$\begin{aligned} D_1^p(\eta,\varphi_0) &= \bigcup_{j=0}^{p-0} \left\{ \left| \arg z - \varphi_0 - \pi\frac{2j}{p} \right| \leqslant \eta \right\}, \\ D_2^p(\eta,\varphi_0) &= \bigcup_{j=0}^{p-1} \left\{ \left| \arg z - \varphi_0 - \pi\frac{2j+1}{p} \right| \leqslant \eta \right\}. \end{aligned} \tag{1}$$

Let

$$a_k = |a_k| e^{i\alpha_k}$$

be the zeros of $f(z)$;

$$b_n = |b_n| e^{i\beta_n}$$

the poles of $f(z)$; and let $\{c_\nu\}$ be the sequence composed of the zeros and poles, counted with their multiplicities.

Definition (cf. ([3])). If there exist φ_0 and η such that

$$\sum_{a_k \in D_1^p(\eta, \varphi_0)} |a_k|^{-p} < \infty, \qquad \sum_{b_n \in D_2^p(\eta, \varphi_0)} |b_n|^{-p} < \infty, \tag{2}$$

then we shall say that the zeros and poles of $f(z)$ are (p, η)-separated.

Obviously, the zeros and poles of $f(z)$ will be (p, η)-separated, in particular, when all $a_k \in D_1^p(\eta, \varphi_0)$ and all $b_n \in D_2^p(\eta, \varphi_0)$. In the case $R < \infty$, condition (2) means that outside the corresponding sets $D_1^p(\eta, \varphi_0)$ and $D_2^p(\eta, \varphi_0)$ there lies only a finite number of zeros and poles.

Theorem 1. Let the zeros and poles of the function $f(z)$ be (p, η)-separated. If a) $R = \infty$ and

$$0 < \limsup \frac{T(r, f)}{r^p} < \infty,$$

then

$$\delta(0) = \delta(\infty) = 1, \qquad d(f) = 2.$$

If b) $R = \infty$ and

$$\limsup \frac{T(r, f)}{r^p} = \infty,$$

or c) $R < \infty$ and

$$\lim T(r, f) = \infty,$$

then

$$d(f) \geqslant \frac{2 \cos p\eta}{1 + \cos p\eta} > 0. \tag{3}$$

Proof. If a) holds, then for the function $f(z)$ with (p, η)-separated zeros and poles the representation $f(z) = f_1(z) \exp P(z)$ is valid, where $f_1(z)$ is a meromorphic function of genus not exceeding $p - 1$, and $P(z)$ is a polynomial of degree p [(3)*]; consequently, $\delta(0, f) = \delta(\infty, f) = 1$ and $d(f) = 2$. For $R = \infty$, from $\limsup T(r, f)/r^p = \infty$ it follows that $\lim T(r, f)/r^p = \infty$ [(3)], and therefore $O(r^p) = o(T(r, f))$. Obviously, the latter is also true for $R < \infty$, since in this case $O(r^p) = O(1)$. Without loss of generality, one may assume $\varphi_0 = 0$. We apply to the circle $|z| \leq r < R$ ($|c_\nu| \neq r$, $\nu = 1, 2, ...$) the Schwarz-Jensen formula ([(1)], p. 165):

$$\ln f(z) = \frac{1}{2\pi} \int_0^{2\pi} \ln |f(re^{i\theta})| \frac{re^{i\theta} + z}{re^{i\theta} - z} \, d\theta +$$

$$+\sum_{|b_n|<r}\ln\frac{r^2-\bar{b}_n z}{r(z-b_n)}-\sum_{|a_k|<r}\ln\frac{r^2-\bar{a}_k z}{r(z-a_k)}+iC,\qquad \operatorname{Im} C=0.$$

Differentiating p times with respect to z, putting $z=0$, and multiplying both sides of the equality obtained by $r^p/(2p!)$, we arrive at the equality

$$\frac{1}{2p!}\,r^p(\ln f(z))^{(p)}\Big|_{z=0}=\frac{1}{2\pi}\int_0^{2\pi}e^{-ip\theta}\ln|f(re^{i\theta})|\,d\theta-$$

$$-\frac{1}{2p}\sum_{|a_k|<r}\frac{r^{2p}-|a_k|^{2p}}{a_k^p r^p}+\frac{1}{2p}\sum_{|b_n|<r}\frac{r^{2p}-|b_n|^{2p}}{b_n^p r^p}.$$

Taking real parts, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\ln|f(re^{i\theta})|\cos p\theta\,d\theta=\frac{1}{p}\sum_{|a_k|<r}\operatorname{sh}\left(p\ln\frac{r}{|a_k|}\right)\cos p\alpha_k-$$

$$-\frac{1}{p}\sum_{|b_n|<r}\operatorname{sh}\left(p\ln\frac{r}{|b_n|}\right)\cos p\beta_n+O(r^p).$$

Taking into account (2), the inequality $0<\operatorname{sh}(p\ln(r/|c_\nu|))\le 2^{-1}(r/|c_\nu|)^p$, $|c_\nu|<r$, and the fact that for $a_k\in D_1^p(\eta,0)$ $\cos p\alpha_k\ge\cos p\eta>0$, and for $b_n\in D_2^p(\eta,0)$ $-\cos p\beta_n\ge\cos p\eta>0$, we obtain

$$\frac{1}{2\pi}\int_0^{2\pi}\ln|f(re^{i\theta})|\cos p\theta\,d\theta\ge\frac{\cos p\eta}{p}\sum_{|c_\nu|<r}\operatorname{sh}\left(p\ln\frac{r}{|c_\nu|}\right)+O(r^p).\qquad(4)$$

The left-hand side of (4) obviously does not exceed $m(r)$, while $\operatorname{sh}(p\ln(r/|c_\nu|))>\rho\ln(r/|c_\nu|)$, and $O(r^p)=o(T(r))$. Therefore $m(r)\ge N(r)\cos p\eta+o(T(r))$. Hence, taking into account $m(r)+N(r)=2T(r)+O(1)$, we easily obtain (3).

In what follows we shall restrict ourselves to the case $R=\infty$, without mentioning this specially. We shall show that if in (1) we put $\eta=\pi/(2p)$ and replace $\le\eta$ by $<\eta$, then, under (2), it may be that $d(f)=0$. Indeed, for $f(z)=B(z^p)$, where

$$B(z)=\prod_{k=1}^{\infty}\frac{1-z/a_k}{1-z/\bar{a}_k},\qquad a_k=|a_k|e^{i\alpha_k}\to\infty,\qquad 0<\alpha_k<\pi,\qquad \sum_{k=1}^{\infty}\frac{\sin\alpha_k}{|a_k|}<\infty,$$

$m(r)=o(r^p)$ outside a certain set of finite logarithmic measure, as follows from a result of Hayman $^{(4)}$**;

---

[*] We take this opportunity to make a correction in $^{(3)}$. In the statements of the theorem and Corollary 3 it is necessary to assert only the existence of the limit $\lim_{r\to\infty} T(r)/r^\lambda$, and not its positivity.

[**] If one assumes $\sum |a_k|^{-1}\sin\alpha_k \cdot \ln(2/\sin\alpha_k) < \infty$, then $m(r) = o(r^p)$ without an exceptional set $^{(5)}$.

At the same time, by choosing $|a_k|$ one can ensure that the characteristic $T(r)$ satisfies conditions a) or b). We also note that, with the help of $B(z)$, using Theorems 2.7 and 2.1 from (^{6}), one can construct examples of meromorphic functions with $(p,0)$-separated zeros and poles for which $T(r)/r^p \to 0$ and $d(f) = 0$.

One can show that, for $0 < \eta < \pi/(2p)$ and under the assumptions of Theorem 1b), it may be that $\delta(0) = \delta(\infty) = 0$. This follows at once from the following assertion. Let ρ be a nonintegral number, $1/2 < \rho < \infty$, and let α and β be arbitrary real numbers. There exists a meromorphic function $f(z)$, in $z \neq \infty$, of order ρ, all of whose zeros lie on the rays $\arg z = \alpha$ and $\arg z = \alpha + \pi/\rho = \alpha_1$, and all of whose poles lie on the rays $\arg z = \beta$ and $\arg z = \beta + \pi/\rho = \beta_1$, for which $\delta(0) = \delta(\infty) = 0$. Let us construct this function. Let $\{p_j\}$ be a monotonically increasing sequence of positive numbers such that $p_{j+1}/p_j \to \infty$ as $j \to \infty$. Let $g_1(z)(g_2(z))$ be the canonical product of genus $[\rho]$ with zeros at all points of the form $n^{1/\rho}$, $n = 1, 2, ...,$ that fall in $[p_{2j-1}, p_{2j})$ (respectively, that fall in $[p_{2j}, p_{2j+1})$), $j = 1, 2, ...$ Denote

$$g(z) = g_1(z)g_2(z).$$

The required function is

$$f(z) = g_1(ze^{-i\alpha})g_1(ze^{-i\alpha_1})\{g_2(ze^{-i\beta})g_2(ze^{-i\beta_1})\}^{-1}.$$

To show that $\delta(0, f) = 0$, write

$$f(z) = g(ze^{-i\alpha})g(ze^{-i\alpha_1})\{g_2(ze^{-i\alpha})g_2(ze^{-i\alpha_1}) \times g_2(ze^{-i\beta})g_2(ze^{-i\beta_1})\}^{-1}.$$

Taking into account the known (^{7}) asymptotic representation for $g(z)$, we obtain that outside disks with centers at the zeros of

$$G(z) = g(ze^{-i\alpha})g(ze^{-i\alpha_1})$$

and with zero linear density, the relation

$$\ln|G(re^{i\varphi})| \sim \frac{\pi}{2}\sin\rho(\varphi-\alpha)\cdot r^\rho \quad \text{for } \varphi \in (\alpha, \alpha_1)$$

is valid, and

$$\ln|G(re^{i\varphi})| = o(r^\rho) \quad \text{for } \varphi \in [\alpha_1, \alpha + 2\pi].$$

On the other hand, one can show that for $r_j \in (p_{2j-1}, p_{2j})$ such that $r_j/p_{2j-1} \to \infty$ and $p_{2j}/r_j \to \infty$ as $j \to \infty$,

$$\ln|g_2(r_{je}^{i\varphi})| = o(r_j^\rho).$$

It follows that $m(r_j, 0, f(z)) = o(r_j^\rho)$, and, since $T(r,f) > Cr^\rho$, $\delta(0,f) = 0$. Similarly one proves that $\delta(\infty, f) = 0$.

For $\eta = 0$ we are not able to construct an analogous example; moreover, it seems probable that the following hypothesis is true. Let the assumptions of Theorem 1b) be satisfied for a function $f(z)$ of finite order, and let $\eta = 0$. Then $\delta(0,f) > 0$ and $\delta(\infty,f) > 0$. We can prove this hypothesis only under certain additional assumptions.

Theorem 2. *Suppose that the assumptions of Theorem 1b) are satisfied and that there exist constants $\lambda > 1$ and $1 > \mu > 0$ such that, for all $r > r_0$,*

$$N(r)/N(\lambda r) \geq \mu.$$

Then

$$d(f) \geq \frac{2\{1 + \varkappa(\lambda)\mu\}\cos p\eta}{1 + \{1 + \varkappa(\lambda)\mu\}\cos p\eta}, \qquad \varkappa(\lambda) = \frac{\operatorname{sh}(p\ln\lambda)}{p\ln\lambda} - 1 > 0. \tag{5}$$

To prove this, note that $\operatorname{sh} x/x$ is a monotonically increasing function of $x \in [0,\infty)$. This makes it possible to estimate the right-hand side in (4) more precisely. Indeed,

$$p^{-1}\sum_{|c_\nu|<r} \operatorname{sh}(p\ln(r/|c_\nu|)) = p^{-1}\sum_{|c_\nu|<r/\lambda} + p^{-1}\sum_{r/\lambda \leq |c_\nu| < r} \geq$$

$$\geq \frac{\operatorname{sh}(p\ln\lambda)}{p\ln\lambda}\sum_{|c_\nu|<r/\lambda} \ln\frac{r}{|c_\nu|} + \sum_{r/\lambda\leq|c_\nu|<r} \ln\frac{r}{|c_\nu|} =$$

$$= \varkappa(\lambda)N(r/\lambda) + N(r) \geq \{\varkappa(\lambda)\mu + 1\}N(r), \qquad r > \lambda r_0.$$

The rest of the argument is carried out as in the proof of Theorem 1.

Corollary 1. *If, under the assumptions of Theorem 2, $\eta = 0$, then $\delta(0,f) > 0$ and $\delta(\infty,f) > 0$.*

Indeed, in this case it follows from (5) that

$$d(f) \geq (2 + 2\varkappa(\lambda)\mu)/(2 + \varkappa(\lambda)\mu) = \psi(\lambda,\mu) > 1,$$

and, consequently,

$$\delta(0,f) \geq \psi(\lambda,\mu) - 1 > 0, \qquad \delta(\infty,f) \geq \psi(\lambda,\mu) - 1 > 0.$$

Corollary 2. Let, for a function $f(z)$ of finite order ρ, the conditions of Theorem 1b) be satisfied, and let $\eta = 0$. Then, for all $r \in [1,\infty)$, except possibly for

a set of upper logarithmic density
$\leq 2\rho \ln \lambda / \ln \mu^{-1}$, $\lambda > 1$, $1 > \mu > 0$, one has

$$m(r, 0) \geq (\psi(\lambda, \mu) - 1)T(r) + o(T(r)),$$

$$m(r, \infty) \geq (\psi(\lambda, \mu) - 1)T(r) + o(T(r)).$$

For the proof it is enough to show that the upper logarithmic density of the set $\mathcal{E}$ (i.e. $\limsup\{$logarithmic measure of $\mathcal{E} \cap (1, r)\}/\ln r$) of those r for which $N(r)/N(\lambda r) < \mu$ does not exceed $2\rho \ln \lambda / \ln \mu^{-1}$, and then to carry out the argument as in the proof of Theorem 2. Of course, Corollary 2 can be of interest only for such a choice of $\lambda > 1$ and $1 > \mu > 0$ that $2\rho \ln \lambda / \ln \mu^{-1} < 1$.

We give one more theorem connected with the topic of the article.

Theorem 2. Let, for an entire function $f(z)$ of genus $\infty > \rho \geq 1$, all zeros lie inside a certain angle with vertex at $z = 0$ and opening, for odd ρ, strictly less than $\pi/(\rho + 1)$, and, for even ρ, strictly less than $\{1 - (2\rho)^{-1}\}\pi/(\rho + 1)$. Then $\delta(0, f) > 0$.

For the case when all zeros of $f(z)$ lie on one ray issuing from $z = 0$, this assertion was proved by Edrei and Fuchs (^{2}). The proof of Theorem 3 is obtained by means of a certain modification of the method of these authors.

Uzhgorod State
University

Received
14 XI 1960

References Cited

1. R. Nevanlinna, *Univalent Analytic Functions*, Moscow-Leningrad, 1941.
2. A. Edrei, W. H. J. Fuchs, Trans. Am. Math. Soc., **93**, 292 (1959).
3. A. A. Gol' dberg, Izv. Vyssh. Uchebn. Zaved., Matem., No. 4, 67 (1960).
4. W. K. Hayman, J. Math. Pura et Appl., **35**, 115 (1956).
5. A. A. Gol' dberg, Ukr. Mat. Zh., **11**, 210 (1959).
6. A. A. Gol' dberg, I. V. Ostrovskii, Scientific Notes of Kharkov State University, **5**, issue 1, 1 (1961).
7. B. Ya. Levin, *Distribution of Zeros of Entire Functions*, Moscow, 1956.