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# MATHEMATICS

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## Abstract

## Full Text

*MATHEMATICS*

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# UNITARY PAIRS OF OPERATORS AND THEIR ANALYTIC CHARACTERISTIC IN THE SPACE $L_2(a, b)$

In the present note, as a natural generalization of the concept of a unitary operator, the concept of a **unitary pair of operators** acting in an abstract Hilbert space is introduced. Then an extension to unitary pairs of operators is given of the well-known theorem of S. Bochner <sup>(1,3)</sup> on the analytic characteristic of unitary operators acting in the Hilbert space of functions  $L_2(a, b)$ . We shall make essential use of this characteristic in our next communication, devoted to certain problems arising in connection with the well-known approximation theorem of Müntz <sup>(4)</sup>.

1°. Let two linear bounded operators  $U_1$  and  $U_2$  be defined in an abstract Hilbert space  $H$ , with ranges  $\Delta_k = U_k H \subseteq H$  ( $k = 1, 2$ ), and let  $U_1^*$  and  $U_2^*$  be the corresponding adjoint operators.

We shall call an ordered pair of operators  $\{U_1, U_2\}$  a **unitary pair** if, for arbitrary elements  $f$  and  $g$  of  $H$ , the conditions

$$(U_1^* f, U_1^* g) = (f, g); \quad (1)$$

$$(U_1 f, U_1 g) + (U_2 f, U_2 g) = (f, g), \quad (2)$$

are satisfied, where  $(f, g)$  is the scalar product generating the metric of the space  $H$ .

We shall denote the class of all such unitary pairs by  $\mathfrak{A}(H)$ . Let us note that if  $\{U_1, U_2\} \in \mathfrak{A}(H)$ , then, generally speaking,  $\{U_2, U_1\} \notin \mathfrak{A}(H)$ .

As is easy to see, conditions (1) and (2) are respectively equivalent to the following:

$$U_1 U_1^* = I; \quad (1')$$

$$U_1^* U_1 + U_2^* U_2 = I, \quad (2')$$

where  $I$  is the identity operator of the space  $H$ .

The following assertions follow immediately from the definition of a unitary pair:

**A.** *In order that a given linear bounded operator  $U_1$  in the space  $H$  be unitary, it is necessary and sufficient that from  $\{U_1, U_2\} \in \mathfrak{A}(H)$  it follow that  $U_2 = O$  (where  $O$  is the operator carrying all elements of  $H$  into the zero element).*

Indeed, the property of unitarity of the given operator  $U_1$  in  $H$  is characterized by the fulfillment of the two conditions

$$U_1^*U_1 = U_1U_1^* = I.$$

But if  $\{U_1, U_2\} \in \mathfrak{A}(H)$ , then, in view of (1') and (2'), these conditions are equivalent to the condition  $U_2^*U_2 = 0$ , i.e. to the condition  $U_2 = 0$ .

**B.** If  $\{U_1, U_2\} \in \mathfrak{A}(H)$ , then the operators

$$P_1 = U_1^*U_1, \quad P_2 = U_2^*U_2 \tag{3}$$

project the space  $H$ , respectively, onto the subspaces  $H_1$  and  $H_2$ , which serve as orthogonal complements of one another.

Indeed, by (1') and (3) we have

$$P_1^2 = P_1, \quad P_1^* = P_1,$$

i.e.  $P_1$  is a projection operator mapping the space  $H$  onto some subspace  $H_1$ . But, by (2') and (3),  $P_2 = I - P_1$ , and therefore  $P_2$  is also a projection operator mapping the space  $H$  onto the subspace  $H_2 = H \ominus H_1$ .

Finally, as mutually orthogonal projection operators,

$$P_1P_2 = P_2P_1 = 0. \tag{4}$$

**C.** If  $\{U_1, U_2\} \in \mathfrak{A}(H)$ , then for any elements  $f$  and  $g$  from  $\Delta_2 = U_2H$  we also have

$$(U_2^*f, U_2^*g) = (f, g). \tag{5}$$

Noting that  $I = P_1 + P_2$  and taking into account (4), for arbitrary elements  $F$  and  $G$  from  $H$  we have:

$$(F, G) = (P_1F + P_2F, P_1G + P_2G) = (P_1F, P_1G) + (P_2F, P_2G).$$

Hence, using the notation (3) and the conditions (1) and (2), we obtain

$$(U_2^*U_2F, U_2^*U_2G) = (U_2F, U_2G),$$

and, since  $f = U_2F \subset \Delta_2$ ,  $g = U_2G \subset \Delta_2$  are arbitrary, our assertion is proved.

**D.** If the linear bounded operator  $U_1$  satisfies condition (1), then there exists a linear bounded operator  $U_2$  (unique among positive and self-adjoint operators) such that  $\{U_1, U_2\} \in \mathfrak{A}(H)$ .

Indeed, under condition (1') the operators

$$P_1 = U_1^*U_1, \quad P_2 = I - P_1$$

are mutually orthogonal projection operators. Consequently,

$$P_2^2 = P_2, \quad P_2^* = P_2,$$

i.e. the operator  $P_2$  is self-adjoint and positive, since

$$(P_2f, f) = (P_2^2f, f) = (P_2f, P_2f) \geq 0.$$

But then, by a known theorem <sup>(2)</sup>, there exists a unique positive self-adjoint square root  $U_2 = P_2^{1/2}$ . The operator  $U_2$  will be the desired one.

Thus, if the operator  $U_1$  is not unitary, then, provided the operator  $U_1^*$  is isometric, it admits a "completion" by means of another operator  $U_2$ , which together with it forms a unitary pair  $\{U_1, U_2\}$ .

2°. The linear system  $L_2(a, b)$  of functions measurable on the interval  $(a, b)$  (where  $-\infty \leq a < b \leq +\infty$ ) and with square-integrable modulus in the metric generated by the scalar product

$$(f, g) = \int_a^b f(x)g(x) dx,$$

is a separable Hilbert space.

Define the function  $e_\xi(x)$ , depending on the parameter  $\xi \neq 0$ , as follows:

$$e_\xi(x) = \begin{cases} 1, & x \in [0, \xi), \\ 0, & x \notin [0, \xi), \end{cases} \quad \xi > 0; \quad e_\xi(x) = \begin{cases} -1, & x \in [\xi, 0), \\ 0, & x \notin [\xi, 0). \end{cases} \quad \xi < 0, \quad (6)$$

and, for simplicity, assume that  $x = 0$  is an interior or boundary point of the interval  $(a, b)$ .

Unitary pairs of operators acting in the space  $L_2(a, b)$  can be characterized analytically by means of the following theorem:

**Theorem 1.** *To any unitary pair of operators  $\{U_1, U_2\} \in \mathfrak{A}(H)$ , acting in the space  $H = L_2(a, b)$ , there correspond four functions*

$$\begin{aligned} K(\xi, x) &= U_1 e_\xi(x), & K^*(\xi, x) &= U_1^* e_\xi(x), \\ R(\xi, x) &= U_2 e_\xi(x), & R^*(\xi, x) &= U_2^* e_\xi(x), \end{aligned} \quad (7)$$

belonging to this space for each fixed  $\xi \in (a, b)$  and possessing the property that the correspondence

$$g_1 = U_1 f, \quad g_2 = U_2 f, \quad f = U_1^* g_1 + U_2^* g_2, \quad f \in H, \quad (8)$$

is realized by means of the formulas

$$\int_0^\xi g_1(x) dx = \int_a^b \overline{K^*(\xi, x)} f(x) dx, \quad \int_0^\xi g_2(x) dx = \int_a^b \overline{R^*(\xi, x)} f(x) dx, \quad (9)$$

$$\int_0^\xi f(x) dx = \int_a^b \overline{K(\xi, x)} g_1(x) dx + \int_a^b \overline{R(\xi, x)} g_2(x) dx, \quad (10)$$

while the correspondence

$$g = U_1^* f, \quad f = U_1 g, \quad f \in H, \quad (11)$$

is realized by means of the formulas

$$\int_0^\xi g(x) dx = \int_a^b \overline{K(\xi, x)} f(x) dx, \quad \int_0^\xi f(x) dx = \int_a^b \overline{K^*(\xi, x)} g(x) dx. \quad (12)$$

Moreover, the functions (7) satisfy the equations:

$$\left. \begin{aligned} \text{a) } & \int_a^b \overline{K(\xi, x)} K(\eta, x) dx + \int_a^b \overline{R(\xi, x)} R(\eta, x) dx \\ \text{b) } & \int_a^b \overline{K^*(\xi, x)} K^*(\eta, x) dx \end{aligned} \right\} = \int_a^b e_\xi(x) e_\eta(x) dx = \begin{cases} \min(|\xi|, |\eta|), & \text{when } \xi\eta \geq 0, \\ 0, & \text{when } \xi\eta \leq 0; \end{cases}$$

$$c) \int_0^\eta K(\xi, x) dx = \int_0^\xi \overline{K^*(\eta, x)} dx;$$

$$d) \int_0^\eta R(\xi, x) dx = \int_0^\xi \overline{R^*(\eta, x)} dx.$$

Conversely, every quadruple of functions

$$K(\xi, x), K^*(\xi, x); R(\xi, x), R^*(\xi, x),$$

satisfying equations a), b), c), d), generates, according to formulas (9)–(10), (12), a certain unitary pair of operators  $\{U_1, U_2\}$ , connected with these functions by formulas (7).

Let us note that, in view of assertion A, Theorem 1 contains Bochner's theorem on the characteristic of unitary operators in  $L_2(a, b)$ . In view of this, each of the functions  $K(\xi, x)$  and  $K^*(\xi, x)$  of Theorem 1 is naturally called the kernel of a unitary operator, if  $U_2 = 0$ , i.e. if

$$R(\xi, x) \equiv R^*(\xi, x) \equiv 0.$$

It is known<sup>3</sup> that if the function  $K^*(\xi, x)$  satisfies equation b) and, moreover, is complete, i.e. from

$$\int_a^b K^*(\xi, x) f(x) dx = 0, \quad \xi \in (a, b),$$

where  $f(x) \in L_2(a, b)$ , it follows that  $f(x) \equiv 0$ , then  $K^*(\xi, x)$  is the kernel of some unitary operator.

When the completeness condition of the kernel  $K^*(\xi, x)$  is removed from assertion , in view of Theorem 1 it follows:

**Theorem 2.** Let the function  $K^*(\xi, x)$  satisfy equation b), i.e.

$$\int_a^b \overline{K^*(\xi, x)} K^*(\eta, x) dx = \int_a^b e_\xi(x) e_\eta(x) dx; \quad \xi, \eta \in (a, b).$$

Then there exists a unique function  $K(\xi, x)$  and functions  $R(\xi, x)$ ,  $R^*(\xi, x)$ , which, together with  $K^*(\xi, x)$ , also satisfy equations a), c), d) of Theorem 1, thereby generating a certain unitary pair of operators  $\{U_1, U_2\}$  in  $L_2(a, b)$  according to formulas (9)–(10), (12).

The functions  $R(\xi, x)$ ,  $R^*(\xi, x)$  are also unique if one requires that

$$R^*(\xi, x) \equiv R(\xi, x).$$

This theorem may be regarded as a fundamental solution of the question of completing incomplete kernels. Realizations of such a completion in some examples important for analysis will be given by us later.

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*Note: Figure translations are in progress. See original paper for figures.*

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