BEHAVIOR OF UNPERTURBED SYSTEMS IN INERTIAL SPACE

1961

SovietRxiv

Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

MECHANICS

V. A. BODNER and V. P. SELEZNEV

BEHAVIOR OF UNPERTURBED SYSTEMS IN INERTIAL SPACE

(Presented by Academician V. S. Kulebakin, 23 I 1961)

Unperturbed systems in three-dimensional inertial space, used as navigation systems, contain three accelerometers mounted on a gyroscopic stabilizer and oriented along the axes of an inertial coordinate system. The accelerometer signals are integrated twice to obtain the coordinates of position. Since the accelerometers do not measure gravitational accelerations, the system provides for three channels of autocompensation of gravitational accelerations. Such systems possess selectivity with respect to accelerations of the center of mass of a body moving in space and can be used to determine linear velocities and coordinates of position. Unperturbed systems with three channels of autocompensation, in contrast to systems with two channels, have many interesting and important features. Some questions of the theory of these systems will be set forth in the present paper.

Consider the behavior of an unperturbed (inertial) navigation system in three-dimensional inertial space. As the coordinate system we take the ecliptic system $x_0y_0z_0$ (Fig. 1), whose origin is located at the center of the Sun, and two axes x_0 or y_0 lie in the plane of the ecliptic. The third axis is perpendicular to this plane. One of the axes x_0 or y_0 may be directed toward the star Spica. The position of an object in such a system will be determined by the rectangular coordinates x, y, z.

Fig. 1

Place on an object moving in three-dimensional space an inertial (unperturbed) navigation system. If F and G are the vectors of external forces and gravitational forces acting on the object, then the components of the absolute accelerations x'', y'', z'' of the object' s center of mass along the axes x_0, y_0, z_0 will be

$$x'' = a_{x0} - g_x, \qquad y'' = a_{y0} - g_y, \qquad z'' = a_{z0} - g_z, \tag{1}$$

where a_{x0}, a_{y0}, a_{z0} are the components of the accelerations caused by external forces, and g_x, g_y, g_z are the components of gravitational accelerations.

Accelerometers whose axes of sensitivity coincide with the axes x_0, y_0, z_0 will measure only the accelerations a_{x0}, a_{y0}, a_{z0}. Owing to drifts of the gyroscopes, the axes of sensitivity of the accelerometers, forming the orthogonal coordinate system xyz, will not be parallel to the axes of the inertial coordinate system $x_0y_0z_0$ (Fig. 2). If we use the fact that the angles α, β, γ are small under real conditions, then for the accelerometer signals one may write

$$a_x = a_{x0} + \Delta a_{xr}, \qquad a_y = a_{y0} + \Delta a_{yr}, \qquad a_z = a_{z0} + \Delta a_{zr}, \tag{2}$$

where $\Delta a_{xr} = a_{y0}\beta - a_{z0}\alpha,\ \Delta a_{yr} = a_{z0}\gamma - a_{x0}\beta,\ \Delta a_{zr} = a_{x0}\alpha - a_{y0}\gamma$.

To form an inertial system with three channels of autocompensation of gravitational accelerations, it is necessary to form compen-

sation signals g_{xk}, g_{yk}, g_{zk}, equal in magnitude and opposite in sign to the components g_x, g_y, g_z. Figure 3 shows the block diagram of an inertial system with three autocompensation channels. From the block diagram we obtain the equations of motion of the system

$$\left[(a_x - g_{xk})\frac{1}{p} + x_0'\right]\frac{1}{p} + x_0 = s_x, \qquad \left[(a_y - g_{yk})\frac{1}{p} + y_0'\right]\frac{1}{p} + y_0 = s_y,$$

$$\left[(a_z - g_{zk})\frac{1}{p} + z_0'\right]\frac{1}{p} + z_0 = s_z, \tag{3}$$

where $x_0, y_0, z_0, x_0', y_0', z_0'$ are the initial values of the object' s coordinates and velocities; s_x, s_y, s_z are the current values of the object' s coordinates, delivered by the inertial system; $p = d/dt$.

The equations (3), using relations (2), may be represented in the form

$$\Delta x'' = \Delta g_x + \Delta a_{xr}, \qquad \Delta y'' = \Delta g_y + \Delta a_{yr},$$

$$\Delta z'' = \Delta g_z + \Delta a_{zr}, \tag{4}$$

where $\Delta x = s_x - x,\ \Delta y = s_y - y,\ \Delta z = s_z - z$ are the errors of the inertial system,

$$\Delta g_x = g_x - g_{xk}, \qquad \Delta g_y = g_y - g_{yk},$$

$$\Delta g_z = g_z - g_{zk} \tag{5}$$

Fig. 2

Figure 2: Fig. 2

Fig. 3

Figure 3: Fig. 3

are the errors characterizing the inaccurate compensation of gravitational accelerations.

Fig. 2

Inaccurate compensation of gravitational accelerations leads to the appearance of errors in the inertial system. These errors enter into the coordinates s_x, s_y, s_z, from which the compensation signals g_{xk}, g_{yk}, g_{zk} are computed. If the inertial system is stable, then the errors $\Delta x, \Delta y, \Delta z$ will decay. Obviously, in an unstable system they will increase.

For the formation of compensation signals it is necessary to know analytic expressions for the acceleration components g_x, g_y, g_z. If m_i, x_i, y_i, z_i $(i = 1, 2, \dots, n)$ are the masses and coordinates of the celestial bodies causing the accelerations of the object, then the components of the accelerations will be

Fig. 3

$$g_x = \sum_{i=1}^{n} f m_i \frac{x - x_i}{R_i^3}, \qquad g_y = \sum_{i=1}^{n} f m_i \frac{y - y_i}{R_i^3}, \qquad g_z = \sum_{i=1}^{n} f m_i \frac{z - z_i}{R_i^3}, \quad (6)$$

where

$$R_i = \sqrt{(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2}, \quad (7)$$

$f = 6.67 \cdot 10^{-8}$ CGS is the gravitational constant.

The computing device must form the compensation signals g_{xk}, g_{yk}, g_{zk} in accordance with formulas (6) and (7), into which, instead of the actual coordinates of the object x, y, z, one should substitute the coordinates s_x, s_y, s_z measured by the inertial system.

Using expressions (5), (6), and (7), we obtain

$$\Delta g_x = -\sum_{i=1}^{n} \frac{f m_i}{R_i^3} \left\{ \left[1 - 3\left(\frac{x - x_i}{R_i} \right)^2 \right] \Delta x - 3 \frac{x - x_i}{R_i^2} \left[(y - y_i)\Delta y + (z - z_i)\Delta z \right] \right\},$$

$$\Delta g_y = -\sum_{i=1}^{n} \frac{fm_i}{R_i^3} \left\{ \left[1 - 3\left(\frac{y - y_i}{R_i}\right)^2 \right] \Delta y - 3\frac{y - y_i}{R_i^2} \left[(x - x_i)\Delta x + (z - z_i)\Delta z \right] \right\}, \tag{8}$$

$$\Delta g_z = -\sum_{i=1}^{n} \frac{fm_i}{R_i^3} \left\{ \left[1 - 3\left(\frac{z - z_i}{R_i}\right)^2 \right] \Delta z - 3\frac{z - z_i}{R_i^2} \left[(x - x_i)\Delta x + (y - y_i)\Delta y \right] \right\}.$$

Substituting (8) into (4), we find

$$(p^2 + \eta_x^2)\Delta x - A\Delta y - B\Delta z = \Delta a_{xr}, \qquad -A\Delta x + (p^2 + \eta_y^2)\Delta y - C\Delta z = \Delta a_{yr},$$

$$-B\Delta x - C\Delta y + (p^2 + \eta_z^2)\Delta z = \Delta a_{zr}, \tag{9}$$

where

$$\eta_x^2 = \sum_{i=1}^{n} \frac{fm_i}{R_i^3} \left[1 - 3\left(\frac{x - x_i}{R_i}\right)^2 \right], \qquad \eta_y^2 = \sum_{i=1}^{n} \frac{fm_i}{R_i^3} \left[1 - 3\left(\frac{y - y_i}{R_i}\right)^2 \right],$$

$$\eta_z^2 = \sum_{i=1}^{n} \frac{fm_i}{R_i^3} \left[1 - 3\left(\frac{z - z_i}{R_i}\right)^2 \right]; \tag{10}$$

$$A = 3\sum_{i=1}^{n} \frac{fm_i}{R_i^3} \frac{(x - x_i)(y - y_i)}{R_i^2}, \qquad B = 3\sum_{i=1}^{n} \frac{fm_i}{R_i^3} \frac{(x - x_i)(z - z_i)}{R_i^2},$$

$$C = 3\sum_{i=1}^{n} \frac{fm_i}{R_i^3} \frac{(y - y_i)(z - z_i)}{R_i^2}. \tag{11}$$

Consequently, the errors of the inertial system caused by inexact autocompensation of accelerations are determined by a system of linear differential equations with variable coefficients.

Let us investigate the stability of the acceleration autocompensation system under the condition that there is no gyroscope drift ($\Delta a_{xr} = \Delta a_{yr} = \Delta a_{zr} = 0$). To this end, let us consider a particular case, assuming that the coefficients of system (9) vary slowly in comparison with the duration of the processes in the autocompensation system. Using this assumption and taking the coefficients of system (9) to be constant (the "frozen-coefficients" method), we can judge its stability on the basis of the characteristic equation

$$p^6 + (\eta_x^2 + \eta_y^2 + \eta_z^2)p^4 + (\eta_x^2\eta_y^2 + \eta_x^2\eta_z^2 + \eta_y^2\eta_z^2 - A^2 - B^2 - C^2)p^2+$$

$$+\eta_x^2\eta_y^2\eta_z^2 - A^2\eta_z^2 - B^2\eta_y^2 - C^2\eta_x^2 - 2ABC = 0. \tag{12}$$

It follows from this that the system for autocompensation of gravitational accelerations is unstable.

Let us transform equation (12), using relations (10) and (11). In doing so, let us assume that the object is in the gravitational field of one equivalent celestial body, at a distance R from the object. The mass of the equivalent celestial body and the distance to it can be found from the condition that the components of the accelerations due to the gravitational forces of the equivalent celestial body be equal to g_x, g_y, and g_z, determined by

by formulas (6). The calculations give

$$\eta_x^2 + \eta_y^2 + \eta_z^2 = 0, \qquad \eta_x^2\eta_y^2 + \eta_x^2\eta_z^2 + \eta_y^2\eta_z^2 - A^2 - B^2 - C^2 = -3\frac{g^2}{R^2}, \tag{13}$$

$$\eta_x^2\eta_y^2\eta_z^2 - A^2\eta_z^2 - B^2\eta_y^2 - C^2\eta_x^2 - 2ABC = -2\frac{g^3}{R^3},$$

where $g = \sqrt{g_x^2 + g_y^2 + g_z^2}$.

Substituting (13) into (12), we obtain

$$(p^2 + \Omega^2)^2(p^2 - 2\Omega^2) = 0, \tag{14}$$

where $\Omega^2 = g/R$.

Consequently, the general motion of the autocompensation system consists of three motions: a harmonic oscillation with period $T_1 = 2\pi\sqrt{R/g}$, an oscillation with the same period and an amplitude increasing proportionally to time, and, finally, an aperiodic exponentially increasing motion with time constant $T_2 = T_1/2\sqrt{2}\pi$.

One may note the analogy between the character of the change in the errors of the inertial system and the motion of a satellite. The period of oscillation of the error $T_1 = 2\pi\sqrt{R/g}$ is equal to the period of revolution of a satellite moving in a circular orbit of radius R around the equivalent celestial body with the first cosmic velocity

$$V_1 = R\Omega = \sqrt{gR}. \tag{15}$$

The time constant of the error $T_2 = T_1/2\sqrt{2}\pi$ is equal to the time constant of the increase of the radius vector R of a satellite receding from the equivalent celestial body with the second cosmic velocity

$$V_2 = \sqrt{2}\Omega R = \sqrt{2gR}. \tag{16}$$

The physical meaning of the indicated analogy is that an inertial system with compensated gravitational accelerations is in conditions analogous to the absence of gravitational forces.

On the basis of the foregoing one may formulate the following theorems.

Theorem 1. Autocompensation of gravitational accelerations in all three channels of an inertial system leads to the characteristic equation $(p^2 + \Omega^2)^2(p^2 - 2\Omega^2) = 0$ for the errors of each of the channels.

Theorem 2. The period of oscillation of the errors of an inertial system is equal to the period of revolution of a satellite moving in a circular orbit of radius R around the equivalent celestial body with the first cosmic velocity $V_1 = \sqrt{gR}$.

Theorem 3. The time constant of the increase of the errors of an inertial system is equal to the time constant of the motion of a satellite receding from the equivalent celestial body with the second cosmic velocity $V_2 = \sqrt{2gR}$.

The instability of inertial systems with three channels of autocompensation of gravitational accelerations cannot serve as a criterion of their unsuitability. The point is that with zero initial conditions and in the absence of gyro drifts the errors will be equal to zero, while in the case of nonzero initial conditions and small gyro drifts they will increase slowly. In this case doubling of the initial errors occurs in a time equal to $0.15T_1$, where T_1 is the period of the system. The system can be made stable by introducing additional external information ([1]).

Received
16 I 1961

CITED LITERATURE

1. V. A. Bodner, V. E. Ovcharov, V. P. Seleznev, DAN, **125**, No. 5 (1959).

Note: Figure translations are in progress. See original paper for figures.