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Abstract

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ON THE PROPAGATION OF RAYLEIGH WAVES ALONG THE SURFACE OF A HOMOGENEOUS ELASTIC BODY OF ARBITRARY SHAPE

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1°. In the present paper, solutions of the equations of the dynamics of an elastic body will be constructed that generalize the well-known solutions of Rayleigh. As is known, "classical" Rayleigh waves may be regarded as the superposition of two complex plane waves—a longitudinal and a transverse one—chosen so that the stresses vanish on the surface of the half-space (^{1}). If the plane wave is replaced by a "ray" solution and the half-space by an arbitrary analytic surface, then we arrive at the generalization of Rayleigh waves that is considered here*.

2°. Let $f_0(\zeta)$ be a function of the complex variable ζ, regular in the upper half-plane, and let $f_k(\zeta)$ be successive integrals of $f_0(\zeta)$. A longitudinal ray solution of the equations of the theory of elasticity is the series (^{2,3})

$$\mathbf{u} = \sum_{k=0}^{+\infty} \mathbf{u}_{ka}(x, y, z)\, f_k(t - \tau_a(x, y, z)), \tag{1}$$

where

$$(\nabla \tau_a)^2 = \frac{1}{a^2} \qquad \left(a^2 = \frac{\lambda + 2\mu}{\rho}\right); \tag{2}$$

$$\mathbf{u}_{ka} = \mathbf{u}^0_{ka} + \varphi_k \nabla \tau_a; \qquad \mathbf{u}^0_{ka} \perp \nabla \tau_a; \qquad \mathbf{u}^0_{ka} = \frac{-\mathbf{M}(\mathbf{u}_{k-1,a}) + \mathbf{L}(\mathbf{u}_{k-2,a})}{\lambda + \mu}\, a^2; \tag{3}$$

$$(\lambda + 2\mu)(2\nabla \varphi_k \nabla \tau_a + \varphi_k \Delta \tau_a) + (\mathbf{M}(\mathbf{u}_{ka}) - \mathbf{L}(\mathbf{u}_{k-1,a}))\nabla \tau_a = 0, \tag{4}$$

$$\mathbf{u}_{-1} = \mathbf{u}_{-2} = 0,$$

where λ and μ are the Lamé parameters,

$$\mathbf{L}(\mathbf{u}) = (\lambda + \mu)\nabla(\nabla\mathbf{u}) + \mu\Delta\mathbf{u}.$$

$\mathbf{M}$ is the operator defined in (^{2}).

A transverse ray solution is the series

$$\mathbf{u}_b = \sum_{k=0}^{+\infty} \mathbf{u}_{kb}(x, y, z)\, f_k(t - \tau_b(x, y, z)), \tag{5}$$

where

$$(\nabla\tau_b)^2 = \frac{1}{b^2}, \qquad b^2 = \frac{\mu}{\rho}; \tag{6}$$

$$\mathbf{u}_{kb} = \mathbf{u}^0_{kb} + \mathbf{u}^1_{kb}; \qquad \mathbf{u}^0_{kb} \parallel \nabla\tau_b; \qquad \mathbf{u}^1_{kb} \perp \nabla\tau_b; \qquad \mathbf{u}^0_{kb} = \frac{\mathbf{M}(\mathbf{u}_{k-1,b}) - \mathbf{L}(\mathbf{u}_{k-2,b})}{\lambda + \mu}\, b^2, \tag{7}$$

the role of condition (4) in this case is played by the requirement that the component of the vector

$$\mathbf{M}(\mathbf{u}^0_{kb} + \mathbf{u}^1_{kb}) - \mathbf{L}(\mathbf{u}_{k-1,b}), \tag{8}$$

perpendicular to $\nabla\tau_b$, vanish. If the series (2) and (5) admit termwise differentiation twice, then the ray solutions satisfy the equations of elasticity theory.

3°. Let the elastic body be bounded by an analytic surface S. Take for τ_a and τ_b solutions of equations (2) and (6) satisfying the conditions

$$\tau_a\big|_S = \tau_b\big|_S = \tau_c; \qquad \operatorname{Im}\frac{\partial\tau_a}{\partial\nu}\bigg|_S < 0; \qquad \operatorname{Im}\frac{\partial\tau_b}{\partial\nu}\bigg|_S < 0; \tag{9}$$

$$\operatorname{Im}\tau_c = 0, \qquad \nabla(\tau_c, \tau_c) = \frac{1}{c^2}. \tag{10}$$

Here ν is the inward normal, $\nabla(\tau_c, \tau_c)$ is the first differential parameter of the function τ_c on the surface S, reducing in the plane case to the square of the derivative with respect to arc length; c is the velocity of Rayleigh waves.

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* In the present paper only the case of a homogeneous medium is considered. By the same method one could also consider the case of an inhomogeneous elastic medium.

Obviously,

$$\frac{\partial\tau_a}{\partial\nu}=(-i)\sqrt{\frac{1}{c^2}-\frac{1}{a^2}},\qquad \frac{\partial\tau_b}{\partial\nu}=(-i)\sqrt{\frac{1}{c^2}-\frac{1}{b^2}}. \tag{11}$$

Introduce on the surface a "semigeodesic" coordinate system (^{4}): through each point of a fixed analytic curve $\mathbf{r}=\mathbf{r}(\alpha)$ on the surface, draw the geodesic line perpendicular to it. Any point M on the surface near the curve $\mathbf{r}=\mathbf{r}(\alpha)$ is characterized by two parameters: $\tau=s/c$, where s is the arc length of the geodesic from its intersection with the curve $\mathbf{r}=\mathbf{r}(\alpha)$ to M, and α, characterizing this point of intersection. Points N outside the surface S are conveniently characterized by the quantity ν—the distance along the normal from N to S—and by the parameters α,τ, characterizing the point M of intersection of the normal passing through N with the surface S.

4°. It is convenient to seek the vector $\mathbf{u}^1_{kb}$ in the form

$$\mathbf{u}^1_{kb}=\psi_k\vec{\xi}+\psi_{k\eta}\vec{\eta}, \tag{12}$$

where

$$\vec{\xi}=-\frac{1}{b\sqrt{1-b^2(\partial\tau_b/\partial\nu)^2}}\left(\vec{\nu}^{\,0}-b^2\nabla\tau_b(\nabla\tau_b\vec{\nu}^{\,0})\right), \tag{13}$$

$$\vec{\eta}=\vec{\alpha}^{\,0}-b^2\nabla\tau_b(\nabla\tau_b\vec{\alpha}^{\,0}), \tag{14}$$

where $\vec{\nu}^{\,0}$ (respectively $\vec{\alpha}^{\,0}$) is the vector which has, in the curvilinear coordinate system τ,α,ν, contravariant components $(0,0,1)$ (respectively $(0,1,0)$).

Obviously, $\vec{\xi}$ and $\vec{\eta}$ are orthogonal to $\nabla\tau_b$, and the length of $\vec{\xi}$ is $1/b$. The condition that vector (8) be parallel to $\nabla\tau_b$ at the points of the surface S is equivalent to the equalities

$$2\nabla\psi_k\nabla\tau_b+\psi_k\Delta\tau_b-\frac{b_{\alpha\tau}}{c}\psi_{k\eta}+\frac{1}{\rho}(\mathbf{M}(\mathbf{u}^0_{kb})-\mathbf{L}(\mathbf{u}_{k-1,b}))\vec{\xi}=0, \tag{15}$$

$$2\nabla\psi_{k\eta}\nabla\tau_b+\psi_{k\eta}\Delta\tau_b+\psi_k\frac{\sqrt{g_{\alpha\alpha}}}{c^2}\left(\frac{\partial\ln\sqrt{g_{\alpha\alpha}}}{\partial\tau}-b_{\alpha\tau}\frac{\partial\tau_b}{\partial\nu}\right)+$$

$$+\frac{c}{b^2}\frac{b_{\alpha\tau}\psi_k}{\sqrt{g_{\alpha\alpha}}c^2}+\frac{1}{\rho}(\mathbf{M}(\mathbf{u}^0_{kb})-\mathbf{L}(\mathbf{u}_{k-1,b}))\vec{\alpha}^{\,0}=0, \tag{16}$$

where ρ is the density. Here and below by $g_{\alpha\alpha},g_{\alpha\tau},g_{\tau\tau}$ (respectively $b_{\alpha\alpha},b_{\alpha\tau},b_{\tau\tau}$) we denote the coefficients of the first (respectively second) quadratic form of Gauss.

5°. Require that the vector $\mathbf{u} = \mathbf{u}_a + \mathbf{u}_b$ satisfy, on the surface S, the condition of absence of stresses:

$$-\lambda(\mathbf{u}_{k+1,a}\nabla\tau_a+\mathbf{u}_{k+1,b}\nabla\tau_b)-2\mu\left(u^{\nu}_{k+1,a}\frac{\partial\tau_a}{\partial\nu}+u^{\nu}_{k+1,b}\frac{\partial\tau_b}{\partial\nu}\right)+\lambda(\nabla\mathbf{u}_{ka}+\nabla\mathbf{u}_{kb})+2\mu\Biggl(\frac{\partial u^{\nu}_{ka}}{\partial\nu}+\frac{\partial u^{k}_{kb}}{\partial\nu}+\Gamma^{\nu}_{\nu\delta}u^{\delta}_{ka}+\Gamma^{\nu}_{\nu\delta} \tag{17}$$

$$-u^{\tau}_{k+1,a}\frac{\partial\tau_a}{\partial\nu}-u^{\tau}_{k+1,b}\frac{\partial\tau_b}{\partial\nu}-u^{\nu}_{k+1,a}\frac{\partial\tau_a}{\partial\tau}-u^{\nu}_{k+1,b}\frac{\partial\tau_b}{\partial\nu}+\frac{\partial u^{\tau}_{ka}}{\partial\nu}+\Gamma^{\tau}_{\delta\nu}u^{\delta}_{ka}+\frac{\partial u^{\tau}_{kb}}{\partial\nu}+\Gamma^{\tau}_{\delta\nu}u^{\delta}_{kb}+\frac{1}{c^2}\Biggl(\frac{\partial u^{\nu}_{ka}}{\partial\tau}+\Gamma^{\nu}_{\tau\delta}u^{\delta}_{ka}+\frac{\partial u^{\nu}_{ka}}{\partial\tau}+$$

(the third equation is not written out for lack of space; $\Gamma^{\nu}_{\tau\delta}$ are the Christoffel symbols of the coordinate system τ, α, ν; summation over repeated indices is understood; u^{τ}, u^{ν} are the contravariant components of the displacement vector). Substituting in (17), instead of $\mathbf{u}_{k+1}$ and $\mathbf{u}_k$, their expressions from formulas (3), (7), (12), we obtain, for finding $\varphi_{k+1}, \psi_{k+1}$, a system of equations with determinant equal to zero ($\psi_{k+1,n}$ is determined uniquely through u^0_{k+1}, u_k from the third equation, not written out here, and the determinant is equal to zero by virtue of Rayleigh' s equation). From the analogous equation for φ_k, ψ_k it follows that φ_k and ψ_k are determined up to terms:

$$\overline{\varphi}_k = e\chi_k, \qquad \overline{\psi}_k = e_2\chi_k, \qquad e_1 = \frac{1}{b^2} - \frac{2}{c^2}, \qquad e_2 = \frac{2i}{c}\sqrt{\frac{1}{c^2} - \frac{1}{a^2}}.$$

Substitute into relation (15)

$$\varphi_k = \varphi^0_k + e_1\chi_k, \qquad \psi_k = \psi^0_k + e_2\chi_k$$

(φ^0_k, ψ^0_k are any values of φ_k and ψ_k found from equations analogous to (17), where k has been replaced by $k-1$). Multiplying the first equation (17) by l_1, the second by l_2, where

$$l_1 = \frac{1}{\mu}\left(\frac{1}{b^2} - \frac{2}{c^2}\right), \qquad l_2 = -2i\sqrt{\frac{1}{c^2} - \frac{1}{b^2}},$$

and adding, we obtain the necessary and sufficient conditions for solvability. They have the form

$$A_1\chi_k + A_2\frac{d\chi_k}{ds} + A_3\frac{\partial\varphi_k}{\partial\nu} + A_4\frac{\partial\psi_k}{\partial\nu} + \ldots = 0$$

(the coefficients A_i are not written out for lack of space).

By dots are denoted terms depending on $\varphi_k^0, \psi_k^0, u_{ka}^a, u_{kb}^0, \mathbf{u}_{k-1,a}, \mathbf{u}_{k-1,b}$. Substituting here the values of $\partial\varphi_k/\partial\nu$ and $\partial\psi_k/\partial\nu$ from formulas (4) and (13), we obtain an ordinary differential equation for χ_k.

Let us write out this equation in the most important case $k = 0$:

$$A\frac{d\chi_0}{ds} + \left(B - \frac{d\ln g_{\alpha\alpha}}{ds} + C\frac{b_{\tau\tau}}{c^2} + D\frac{b_{\alpha\alpha}}{g_{\alpha\alpha}}\right)\chi_0 = 0. \tag{18}$$

Here

$$A = -\frac{8}{c}\left(\frac{1}{b^2} - \frac{2}{c^2}\right)^2 + \frac{4}{c}\left(\frac{1}{b^2} - \frac{2}{c^2}\right)\left(\frac{1}{a^2} - \frac{2}{c^2}\right)\frac{\sqrt{1/c^2 - 1/b^2}}{\sqrt{1/c^2 - 1/a^2}} + \frac{4}{c}\left(\frac{1}{b^2} - \frac{2}{c^2}\right)^2\frac{\sqrt{1/c^2 - 1/a^2}}{\sqrt{1/c^2 - 1/a^2}}, \qquad B = \frac{1}{4}A; \tag{19}$$

$$C = \left(\frac{1}{b^2} - \frac{2}{c^2}\right)\frac{C_1\partial\tau_a/\partial\nu + C_2\partial\tau_b/\partial\nu}{(1/c^2 - 1/a^2)(1/c^2 - 1/b^2)}, \tag{20}$$

where

$$C_1 = 2\left(\frac{1}{c^2} - \frac{1}{a^2}\right)\left(\frac{1}{c^2b^2} - \frac{1}{b^4} - \frac{1}{c^4}\right),$$

$$C_2 = \left(\frac{1}{b^2} - \frac{2}{c^2}\right)\left(\frac{1}{c^2} - \frac{1}{a^2}\right)\left(\frac{1}{c^2} - \frac{3}{2b^2}\right) + \frac{2}{c^2}\left(\frac{1}{c^2} - \frac{1}{b^2}\right)\left(\frac{2}{c^2} - \frac{3}{a^2}\right), \tag{21}$$

$$D = \left(\frac{1}{b^2} - \frac{2}{c^2}\right)\cdot 2\cdot\left(\frac{1}{b^2} - \frac{b^2}{c^4}\right)\frac{\partial\tau_a}{\partial\nu} + \left(\frac{1}{b^2} - \frac{2}{c^2}\right)\frac{b^2}{2}\left(\frac{1}{b^4} + \frac{4}{c^4}\right)\frac{\partial\tau_b}{\partial\nu}.$$

It is not difficult to show that $A > 0$. The coefficients C and D are purely imaginary, and therefore

$$|\chi_0(M)| = |\chi_0(M_0)|\sqrt{\frac{J(M_0)}{J(M)}}, \tag{22}$$

where $J(M) = \sqrt{g_{\alpha\alpha}}$ are the coefficients of divergence of the surface rays, i.e., of the geodesic lines. It is natural to call $|\chi_0(M)|$ the intensity of the Rayleigh waves. Formula (22) is analogous to the usual formulas of the ray method ([2,3]).

In the plane case the terms with B and D drop out, $b_{\tau\tau}/c^2$ should be replaced by $1/R$, where R is the radius of curvature of the boundary. It follows from formula (22) that in the plane case the intensity of the Rayleigh wave does not change in the course of its propagation, as was naturally to be expected.

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Note: Figure translations are in progress. See original paper for figures.