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N. S. SINYUKOV

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Abstract

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N. S. SINYUKOV

ON AN INVARIANT TRANSFORMATION OF RIEMANNIAN SPACES WITH COMMON GEODESICS

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In the paper, first any two Riemannian spaces with common geodesics are considered. From the metric tensors of these spaces and the function determining their geodesic mapping, the metric tensors of another pair of Riemannian spaces, also with common geodesics, are constructed. Then the spaces obtained by applying this transformation to spaces of constant curvature and, generally speaking, not themselves spaces of constant curvature, are considered. Finally, for somewhat more general spaces, the existence is proved of a nontrivial geodesic mapping onto Riemannian spaces with a metric tensor of a specified form.

1. Let the Riemannian space V_n with metric tensor g_{ij} $(i, j = 1, 2, \ldots, n)$ admit a nontrivial geodesic mapping onto the Riemannian space $\overline{V}_n$ with metric tensor $\overline{g}_{ij}$, determined by the gradient $\psi_i \neq 0$, i.e.

$$\overline{g}_{ij,k} = 2\psi_k \overline{g}_{ij} + \psi_i \overline{g}_{kj} + \psi_j \overline{g}_{ki}, \tag{1}$$

where the comma denotes covariant differentiation in V_n. Setting

$$\breve{g}_{ij} = e^{2\psi} \overline{g}^{\alpha\beta} g_{\alpha i} g_{\beta j}, \tag{2}$$

where $\overline{g}^{ij}$ are the elements of the inverse matrix to $\|\overline{g}_{ij}\|$, from (1) we obtain

$$\breve{g}_{ij,k} = \lambda_i g_{kj} + \lambda_j g_{ki}, \qquad \lambda^i = -\psi_\alpha \overline{g}^{\alpha i} e^{2\psi}. \tag{3}$$

It is easy to see that the reverse passage from (3) to (1) is always possible if $|\breve{g}_{ij}| \neq 0$. Equations (3) show that the tensor $\breve{g}_{ij}$ is determined from them by λ_i and g_{kj} up to a covariantly constant tensor in V_n. Consequently, the problem of geodesic mapping of V_n is also solved up to covariantly constant tensors. This fact, in a special case, was found earlier in ([1]). Introducing the tensor

$$\hat{g}_{ij} = e^{2\varphi} \breve{g}_{ij}, \tag{4}$$

from (3) we obtain in $\check{V}_n$, with metric tensor $\check{g}_{ij}$, equations of the form (1).

Thus, the following has been proved.

Theorem 1. *If the Riemannian space V_n with metric tensor g_{ij} admits a nontrivial geodesic mapping onto $\overline{V}_n$ with metric tensor $\overline{g}_{ij}$, determined by the gradient ψ_i, then the Riemannian space $\check{V}_n$ admits a nontrivial geodesic mapping onto $\hat{V}_n$, determined*

by the same vector ψ_i, and their metric tensors are determined by formulas (2) and (4), respectively.

The transformation thus found of a pair of Riemannian spaces with common geodesics will, for brevity, be called a Γ-transformation. Since under the conditions of the theorem $\bar{V}_n$ admits a geodesic mapping onto V_n, determined by the vector $-\psi_i$, we similarly obtain one more pair of spaces with common geodesics.

Verification shows that application of a Γ-transformation to the spaces $\check{V}_n$ and $\hat{V}_n$ returns us to V_n and $\bar{V}_n$, i.e., the Γ-transformation is involutive in character.

2. Let us consider the Γ-transformation as applied to spaces V_n of constant curvature K and $\bar{V}_n$ of constant curvature $\bar{K}$ $(n > 2)$. Then, in addition to equations (1), we have the condition

$$Kg_{ij} - \psi_{ij} = \bar{K}\bar{g}_{ij}, \qquad \psi_{ij} = \psi_{i,j} - \psi_i\psi_j. \tag{5}$$

Using the relation between the curvature tensors of the conformal spaces $V_n, \hat{V}_n$ and relation (5), we obtain

$$\hat{R}_{ij} = Ag_{ij} + B\bar{g}_{ij}, \qquad A = (n-1)(K + \Delta_1\psi) - \bar{K}\bar{g}, \qquad B = -\bar{K}(n-2), \tag{6}$$

where $\hat{R}_{ij}$ is the Ricci tensor of $\hat{V}_n$; $\Delta_1\psi$ is the first differential parameter of Beltrami; $\bar{g} = g^{\alpha\beta}\bar{g}_{\alpha\beta}$. But if $\hat{V}_n$ is also a space of constant curvature $\hat{K}$, then from (6) it follows that

$$\left[-(n-1)\hat{K}e^{2\psi} - A\right] g_{ij} = B\bar{g}_{ij},$$

and, for a nontrivial geodesic mapping, this is possible only when $B = 0$, or, what is the same, when $\bar{K} = 0$. It is not difficult to see that this condition is also sufficient for $\hat{V}_n$ to be a space of constant curvature. Thus we have proved:

Theorem 2. *In order that $\hat{V}_n$ be a space of constant curvature, it is necessary and sufficient that $\bar{V}_n$ be a flat space.*

Denote by L_n^1 the class of all Riemannian spaces $\hat{V}_n$ obtained from spaces of constant curvature by a Γ-transformation.

3. Joint consideration of equations (1) and (6) gives the following condition on spaces of class L_n^1:

$$\hat{R}_{ij\Lambda K} = \sigma_k \hat{g}_{ij} + \varphi_i \hat{g}_{kj} + \varphi_j \hat{g}_{ki}, \tag{7}$$

where we have set

$$\sigma_k = e^{-2\psi}(A_{,k} - 2A\psi_k), \qquad \varphi_i = e^{2\psi}(\psi^\alpha \hat{R}_{\alpha i} - A\psi_i), \qquad \psi^\alpha = \psi_\beta g^{\beta\alpha}, \tag{8}$$

and directly from (7) it follows that

$$\sigma_i = N_1 \hat{R}_{,i}, \qquad \varphi_i = N_2 \hat{R}_{,i}, \qquad N_1 = \frac{n}{n(n+1)-2}, \qquad N_2 = \frac{n-2}{2[n(n+1)-2]}. \tag{9}$$

Calculations show that in $\hat{V}_n$ the tensor $\breve{g}_{ij}$, determined by the equalities (2) and (6), as a consequence of (7) and (8) satisfies equations of the form (1). At the same time, from (5) and (6) we have

$$\psi_{ij} = \left(K + \frac{A}{n-2}\right) e^{-2\psi} \hat{g}_{ij} - \frac{1}{n-2} \hat{R}_{ij}. \tag{10}$$

These equations may be regarded as conditions in $\hat{V}_n$ on ψ and A. Now it is not difficult to verify that V_n, determined by formula (4), will be a space-

with constant curvature, if $\hat{V}_n$ is conformally flat and ψ_i satisfies condition (10).

Thus, we have:

Theorem 3. *In order that $\hat{V}_n$ belong to the class L_n^1, it is necessary and sufficient that it be conformally flat, satisfy condition (7), and that there exist a nontrivial solution of the system of equations (8), (10) with respect to ψ_i and A.*

4. Finally, let us consider Riemannian spaces $\hat{V}_n$ satisfying only relation (7). We shall denote this class of spaces by L_n^2. Putting

$$\rho_{ij} = \hat{R}_{ij} - \hat{\sigma} \hat{g}_{ij} + \alpha \hat{g}_{ij}, \quad (\alpha = \text{const}), \tag{11}$$

we see that the tensor ρ_{ij} satisfies equations of the form (3), and, moreover, because of the arbitrariness of α, one may always assume $|\rho_{ij}| \neq 0$. But then, as was indicated earlier, a transition is possible to relations of the form (1) in $\hat{V}_n$, which will be satisfied by the tensor

$$\bar{\rho}_{ij} = e^{2\psi}\rho^{\alpha\beta}\hat{g}_{\alpha i}\hat{g}_{\beta j}, \tag{12}$$

where $\psi_i = -\varphi^\gamma \rho^{\alpha\beta}\hat{g}_{\alpha\gamma}\hat{g}_{\beta i}$, and ρ^{ij} are the elements of the inverse matrix for $\|\rho_{ij}\|$.

Taking (9) into account, we obtain the theorem:

Theorem 4. *Every Riemannian space $\hat{V}_n$ of the class L_n^2 $(n > 2)$ with nonconstant scalar curvature admits a nontrivial geodesic mapping onto a Riemannian space with metric tensor $\bar{\rho}_{ij}$, determined by equalities (11), (12).*

Equations (7) and (9) give a tensorial characteristic of Riemannian spaces of the class L_n^2.

Odessa State University
named after I. I. Mechnikov

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1. N. S. Sinyukov, DAN, 111, No. 4 (1956).

Note: Figure translations are in progress. See original paper for figures.