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# MATHEMATICS

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**Abstract**

**Full Text**

MATHEMATICS

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## ON A GEOMETRIC PROBLEM IN APPROXIMATION THEORY

*(Presented by Academician A. N. Kolmogorov on 27 IV 1961)*

**1. General formulation of the question.** Consider a metric space  $R$ , an arbitrary set  $G$  in it, some bounded set  $F$ , and the quantity

$$\mathcal{E}_G(F) = \sup_{f \in F} \inf_{g \in G} \rho(f, g), \quad (1)$$

which is called the deviation of  $F$  from  $G$ .

In the constructive theory of functions, various versions of problems connected with determining the upper bound  $\mathcal{E}_G(F)$  are well known. In these problems  $R$  is usually a linear normed space. At present a number of exact or asymptotically exact results are known for the case when  $G$  is a finite-dimensional subspace of  $R$  (the case of approximation by "polynomials" of a given order). Some estimates of this kind are also available in other cases. We shall cite one well-known result of M. G. Krein, corresponding to the space of functions  $f(t)$  that are bounded and uniformly continuous on the entire real axis, where  $\rho(f, g) = \sup_{-\infty < t < \infty} |f(t) - g(t)|$ .

If  $F$  consists of all  $r$ -times differentiable functions for which  $\sup_{-\infty < t < \infty} |f^{(r)}(t)| \leq M$ , and  $G$  is the set of all entire functions of degree  $\leq \sigma$ , then (see <sup>(1,7)</sup>)

$$\mathcal{E}_G(F) = \frac{4M}{\pi\sigma^r} \sum_{k=0}^{\infty} \frac{(-1)^{k(r+1)}}{(2k+1)^{r+1}}. \quad (2)$$

In the present note the quantity (1) is considered for some other sets\*.

**2. An elementary example for the plane.** Let  $R_2$  be the two-dimensional real plane, where the distance between the points  $P(x', y')$  and  $Q(x'', y'')$  is defined by the formula  $\rho(P, Q) = \max\{|x' - x''|, |y' - y''|\}$ ; let  $F_b$  be the set of all points whose coordinates satisfy the condition  $|x - y| \leq b$ , where  $c < \infty$  is some positive constant. If  $F$  is the intersection of  $F_b$  with the square  $|x| \leq c$ ,  $|y| \leq c$ , and  $G$  is the intersection of this square with  $F_a$ , then it is obvious that for  $0 \leq a \leq b \leq c$  the equality

$$\mathcal{E}_G^{R_2}(F) = \frac{1}{2}(b - a). \quad (3)$$

holds.

**3. An analogous example for  $m$ -dimensional space.** Consider the  $m$ -dimensional real space  $R_m$ , where the distance between the points  $P(x'_1, \dots, x'_m)$  and  $Q(x''_1, \dots, x''_m)$  is defined by the formula  $\rho(P, Q) = \max_{1 \leq k \leq m} |x'_k - x''_k|$ . Let  $B$  and  $A$  be, respectively—

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\* The contents of this note were presented in a report by the author at the seminar on the theory of functions at Dnepropetrovsk University on March 28, 1961.

respectively, two symmetric square matrices of  $m^2$  real numbers  $\{b_{\nu k}\}$  and  $\{a_{\nu k}\}$  such that  $b_{\nu\nu} = a_{\nu\nu} = 0$ ,  $a_{\nu k} > 0$  ( $\nu \neq k$ ),  $a_{\nu k} \leq a_{\nu s} + a_{ks}$ ,  $b_{\nu k} > 0$  ( $\nu \neq k$ ),  $b_{\nu k} \leq b_{\nu s} + b_{ks}$ , and  $c < \infty$  is a certain constant satisfying the condition  $\max_{1 \leq \nu, k \leq m} a_{\nu k} \leq c$ ,  $\max_{1 \leq \nu, k \leq m} b_{\nu k} \leq c$ . Denote by  $F_{b_{\nu k}}$  the set of all points  $R_m$  for which  $|x_\nu - x_k| \leq b_{\nu k}$ , and by  $F_B$  the intersection of all  $F_{b_{\nu k}}$  ( $\nu, k = 1, \dots, m$ ). If  $F$  is the intersection of  $F_B$  with the  $m$ -dimensional “sphere” of radius  $c$  whose center is at zero, and  $G$  is the intersection of this sphere with  $F_A$ , then for the polyhedra  $F$  and  $G$  the equality holds

$$\mathcal{E}_G^{R_m}(F) = \frac{1}{2} \max_{1 \leq \nu, k \leq m} (b_{\nu k} - a_{\nu k}). \quad (4)$$

**4. Best uniform approximation of functions in a prescribed system of points.** Let a system of points  $t_1 < \dots < t_m$  be given. The matrices  $B$  and  $A$  considered above determine the class  $F$  of all real functions  $f(t)$  satisfying the conditions  $|f(t_\nu) - f(t_k)| \leq b_{\nu k}$  ( $\nu, k = 1, \dots, m$ ), and the class  $G$  of all real functions  $g(t)$  satisfying the conditions  $|g(t_\nu) - g(t_k)| \leq a_{\nu k}$  ( $\nu, k = 1, \dots, m$ ). If we put  $\rho(f, g) = \max_{1 \leq k \leq m} |f(t_k) - g(t_k)|$ , then in this case the quantity  $\mathcal{E}_G(F)$  will coincide with the right-hand side of (4).

**5. Classes of functions with a prescribed majorant of the modulus of continuity.** Consider now the space  $C$  of all functions  $f(t)$  continuous on the interval  $[a, b]$ , where  $\rho(f, g) = \sup_{a \leq t \leq b} |f(t) - g(t)|$ . Let  $\omega(u)$  be an arbitrary modulus of continuity, i.e., a continuous nondecreasing and subadditive function equal to zero for  $u = 0$  (see (7)). By  $H_\omega$ , as usual, denote the set of all real continuous functions  $f(t)$  satisfying the condition  $|f(t) - f(\tau)| \leq \omega(|t - \tau|)$  for any two points  $t$  and  $\tau$  in  $[a, b]$ . In view of what was said in item 4, whatever the moduli of continuity  $\omega_1(u)$  and  $\omega_2(u)$ , for any system of points  $a = t_1 < \dots < t_{m+1} = b$  and for any function  $f(t)$  for which  $|f(t_\nu) - f(t_k)| \leq \omega_1(|t_\nu - t_k|)$ , there exists a function  $g(t)$  such that  $|g(t_\nu) - g(t_k)| \leq \omega_2(|t_\nu - t_k|)$  and

$$\max_{1 \leq k \leq m+1} |f(t_k) - g(t_k)| \leq \frac{1}{2} \max_{0 \leq u \leq b-a} \{\omega_1(u) - \omega_2(u)\}. \quad (5)$$

Successively supplementing the system of points  $t_k$  in the corresponding way (for example, choosing  $m = 2^p$  and  $t_k = a + \frac{k-1}{2^p}(b-a)$ ) and applying the diagonal process, after passing to the limit, by compactness of the class  $H_{\omega_2}$ , we obtain from this that there exists a function  $g_*(t) \in H_{\omega_2}$  satisfying the inequality

$$\rho(f, g_*) \leq \frac{1}{2} \max_{0 \leq u \leq b-a} \{\omega_1(u) - \omega_2(u)\}. \quad (6)$$

The last inequality cannot be improved.

Thus, the following holds.

**Theorem 1.** Whatever the moduli of continuity  $\omega_1(u)$ ,  $\omega_2(u)$  in the space  $C$  of functions continuous on the interval  $[a, b]$ , the equality

$$\mathcal{E}_{H_{\omega_2}}^C(H_{\omega_1}) = \frac{1}{2} \max_{0 \leq u \leq b-a} \{\omega_1(u) - \omega_2(u)\} \quad (7)$$

always holds.

Considering the space  $C^*$  of all continuous  $2\pi$ -periodic functions and the corresponding classes  $H_{\omega_1}^*$  and  $H_{\omega_2}^*$ , from the same considerations we obtain:

**Theorem 2.** Whatever the moduli of continuity  $\omega_1(u)$ ,  $\omega_2(u)$ , in the space  $C^*$  the equality

$$\mathcal{E}_{H_{\omega_2}^*}^{C^*}(H_{\omega_1}^*) = \frac{1}{2} \max_{0 \leq u \leq \pi} \{\omega_1(u) - \omega_2(u)\}. \quad (8)$$

always holds.

If, for example,  $\omega_1(t) = Mt^\alpha$  ( $0 < \alpha \leq 1$ ), and  $\omega_2(t) = Kt^\alpha$  ( $K < M$ ), then

$$\mathcal{E}_{H_{\omega_2}^*}^{C^*}(H_{\omega_1}^*) = \frac{\pi^\alpha}{2}(M - K).$$

**Theorem 3.** Let  $Z_1$  be the class of all functions  $f(t)$  defined on  $[-1, 1]$  and satisfying the conditions  $f(+1) = f(-1) = 0$ ,  $|f(t_1) - f(t_2)| \leq 1$ ,

$$\left| f(t_1) - 2f\left(\frac{t_1 + t_2}{2}\right) + f(t_2) \right| \leq |t_1 - t_2|, \quad t_1, t_2 \in [-1, 1].$$

Then in the space  $C$  of all functions continuous on  $[-1, 1]$ , for  $\omega(u) = mu$  ( $m$  an integer), the equality

$$\mathcal{E}_{H\omega}^C(Z_1) = \frac{1}{2^{m+1}}$$

holds.

The proof may be obtained with the aid of Theorem 1 from (6).

**6. One application.** For applications of equalities (7) and (8) the following obvious relation is useful:

$$\inf_M \left\{ \max_{0 \leq u \leq b-a} [\omega_1(u) - M\omega_2(u)] + M\omega_2(h) \right\} = \omega_1(h), \quad (9)$$

valid for any positive  $h \leq b - a$ , under the condition that  $\omega_1(u)$  and  $\omega_2(u)$  are arbitrary convex moduli of continuity for which, for every  $M \geq 0$ , the difference  $\omega_1(u) - M\omega_2(u)$  is either monotone on  $[0, b - a]$ , or has an increment which, as  $u$  increases, changes sign once from plus to minus. In particular, equality (9) is always valid for  $\omega_2(u) = u$ . An example of Cantor's singular function, which is a modulus of continuity (see (7), § 3.2.4), shows that even in the case  $\omega_2(u) = u$ , without the assumption of convexity of the function  $\omega_1(u)$ , relation (9) may already fail.

Let us note one such application, the idea of which belongs to Lebesgue (9) (see also (2), p. 238) and occurs in (1) (p. 215). By virtue of equality (8) for  $\omega_2(u) = Mu$  and relation (9) for  $h = \pi/n$ , relying on the well-known result of N. I. Akhiezer—M. G. Krein and J. Favard (7), we have that the best uniform approximation by trigonometric polynomials of order  $\leq n - 1$  of any  $2\pi$ -periodic function  $f(t) \in H_{\omega_1}^*$  does not exceed  $\frac{1}{2}\omega_1(\pi/n)$ . The latter estimate, as the example of the  $2\pi/n$ -periodic function

$$f(t) = \begin{cases} \omega_1(t), & 0 \leq t \leq \frac{\pi}{n}, \\ \omega_1\left(\frac{2\pi}{n} - t\right), & \frac{\pi}{n} \leq t \leq \frac{2\pi}{n}, \end{cases} \quad (10)$$

shows, in the general case cannot be improved. In the case  $\omega_1(u) = u^\alpha$  ( $0 < \alpha < 1$ ), N. P. Korneichuk arrived at it, relying on considerations connected with the theorems of P. L. Chebyshev and Vallee-Poussin on alternation.\*

In view of the fact that the function (10) depends on  $n$ , we give the following proposition:

**Theorem 4.** For any convex modulus of continuity  $\omega_1(u)$

$$\sup_{f \in H_{\omega_1}^*} \overline{\lim}_{n \rightarrow \infty} \frac{E_{n-1}^*(f)}{\omega_1\left(\frac{\pi}{n}\right)} = \frac{1}{2}. \quad (11)$$

\* Communication at the seminar on function theory at Dnepropetrovsk University, March 21 and 28, 1961.

where

$$E_{n-1}^*(f) = \inf_{a_k, b_k} \max_t \left| f(t) - \sum_{k=0}^{n-1} (a_k \cos kt + b_k \sin kt) \right|.$$

For the proof of Theorem 4 one may use a result of S. M. Nikol'skii<sup>5</sup> on the upper bound for the approximation of functions of the class  $H_\omega^*$  by interpolating trigonometric polynomials with equidistant nodes.

**Theorem 5.** If  $\omega_1(u)$  is an arbitrary convex modulus of continuity, and  $D_{2n-1}(H_{\omega_1}^*)$  is the width of order  $2n-1$  (see (3, 8)) for  $H_{\omega_1}^*$ , then

$$D_{2n-1}(H_{\omega_1}^*) = \frac{1}{2} \omega_1\left(\frac{\pi}{n}\right). \quad (12)$$

This theorem supplements a recent result of Lorentz<sup>4</sup>.

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*Note: Figure translations are in progress. See original paper for figures.*

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