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Abstract

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MATHEMATICS

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ON THE THEORY OF SINGULAR INTEGRAL EQUATIONS SUBJECT TO THE FREDHOLM ALTERNATIVE

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Let, on a simple closed Lyapunov contour L, there be given a function $\alpha(t)$, having a derivative $\alpha'(t)$ satisfying on L a Hölder condition ($\alpha'(t) \subset H$) and different from zero at the points of L. Suppose that the function $\alpha(t)$ effects a one-to-one and orientation-preserving mapping of the contour L onto itself. The contour L divides the plane into two domains: a finite domain D^+, containing the origin, and an infinite domain D^-.

Consider the singular integral equation

$$\alpha(t)\overline{\varphi(t)} + \frac{b(t)}{\pi i}\int_L \frac{\varphi(\tau)}{\tau - \alpha(t)}\, d\tau = c(t), \tag{1}$$

in which the functions $a(t), b(t), c(t) \subset H$ on L, and $a(t) \neq 0,\ b(t) \neq 0$ at the points of L. If $\alpha(t) \equiv t$, then equation (1) takes the form

$$a(t)\overline{\varphi(t)} + \frac{b(t)}{\pi i}\int_L \frac{\varphi(\tau)}{\tau - t}\, d\tau = c(t). \tag{2}$$

When the conditions

$$\alpha[\alpha(t)] = t, \tag{3}$$

$$a(t)a[\overline{\alpha(t)}] = b(t)b[\overline{\alpha(t)}], \tag{4}$$

$$|a(t)| = |b(t)|, \quad \text{if } \alpha(t) \equiv t, \tag{5}$$

are fulfilled on L, it proves possible to give a complete qualitative investigation of the singular integral equations (1) and (2).

It is established that for equations (1) and (2), as also for the singular integral equation considered in (1), the Fredholm alternative is valid.

Using the well-known method of continuation into the complex plane by means of an integral of Cauchy type (see, for example, (2)), we reduce the integral equations (1) and (2) to systems of boundary-value problems of the type of Carleman’ s problem (3,4). Introduce the function

$$\Phi(z) = \frac{1}{2\pi i} \int_L \frac{\varphi(\tau)}{\tau - z}\, d\tau, \tag{6}$$

where the density $\varphi(t)$ is the desired solution of equation (1) or (2). Using the Sokhotski formulas for the limiting values of the integral (6), we reduce equation (1) to a boundary-value problem for the piecewise-analytic function $\Phi(z)$:

$$a(t)\overline{\Phi^+(t)} + b(t)\Phi^+[\alpha(t)] - a(t)\overline{\Phi^-(t)} + b(t)\Phi^-[\alpha(t)] = c(t). \tag{7}$$

The boundary-value problem corresponding to equation (2) is obtained from (7) if in the boundary condition (7) one sets $\alpha(t) \equiv t$. Using conditions (3)–

(5), we reduce problem (7) to the equivalent (7) system of boundary-value problems

$$\Phi^+[a(t)] = G(t)\overline{\Phi^+(t)} + g_+(t), \tag{8}$$

$$\Phi^-[a(t)] = -G(t)\overline{\Phi^-(t)} + g_-(t), \tag{9}$$

where

$$G(t) = -\frac{a(t)}{b(t)}, \qquad g_\pm(t) = \frac{c(t)\overline{a[\alpha(t)]} \pm b(t)c[\alpha(t)]}{2a[\alpha(t)]b(t)}.$$

Let us note that conditions (3)–(5) ensure the fulfillment of the necessary solvability conditions for problems (8) and (9),

$$G[a(t)]\overline{G(t)} = 1, \qquad G[a(t)]\overline{g_\pm(t)} + g_\pm[a(t)] = 0 \quad \text{on } L \tag{10}$$

for an arbitrary function $c(t) \subset H$ on L.

Denote $\varkappa = \operatorname{Ind} G(t) = \frac{1}{2\pi}\{\arg G(t)\}_L$.

The study of the boundary-value problems (8) and (9), which are also of independent interest, is based on the following basic propositions.

Lemma 1. The boundary-value problem

$$\Phi[a(t)] - \lambda\overline{\Phi(t)} = 0 \quad \text{on } L, \qquad \lambda = \pm 1$$

has as its solution a purely real (for $\lambda = 1$) or purely imaginary (for $\lambda = -1$) constant.

Lemma 2. A transformation $t_1 = \alpha(t)$ of the contour L onto itself, preserving the direction of traversal on L and satisfying condition (3), either has no fixed point on L, or else $\alpha(t) \equiv t$. In the case $\alpha(t) \not\equiv t$, the index $G(t)$ can only be an even number.

The proof of the lemma follows easily from condition (3) and the first condition (10).

At the basis of the study of the boundary-value problems (8) and (9) in the case when $\alpha(t) \equiv t$ and the index $G(t)$ is odd ($\operatorname{Ind} G(t) = 2n - 1$) lies:

Lemma 3. The boundary-value problems on L

$$\Phi^{\pm}(t) - \lambda \frac{|t|}{t} \overline{\Phi^{\pm}(t)} = 0,$$

where $\lambda = \pm 1$, and the function $\Phi^{-}(z)$ satisfies the additional condition $\Phi^{-}(\infty) = 0$, have no nontrivial solutions.

We formulate the principal results of the study of the boundary-value problems (8) and (9) under assumptions (10).

For $\varkappa = 2n \geq 0$, problem (8) is unconditionally solvable; its general solution depends linearly on $\varkappa + 1$ arbitrary real constants and is given by the formula

$$\Phi^{+}(z) = z^{n} X_0^{+}(z) \left[\frac{1}{2\pi i} \int_L \frac{\varphi[\alpha(\tau)]}{\tau - z}\, d\tau + \sum_{j=0}^{2n} \beta_j w_j^{+}(z) \right], \tag{11}$$

where

$$w_0^{+}(z) = 1, \qquad w_{2k-1}^{+}(z) = z^{-k} \frac{1}{2\pi i} \int_L \frac{\varphi_{2k-1}[\alpha(\tau)]}{\tau - z}\, d\tau,$$

$$w_{2k}^{+}(z) = i z^{-k} + \frac{1}{2\pi i} \int_L \frac{\varphi_{2k}[\alpha(\tau)]}{\tau - z}\, d\tau, \qquad X_0(z) = \exp\left\{ \frac{1}{2\pi i} \int_L \frac{\gamma_0[\alpha(\tau)]}{\tau - z}\, d\tau \right\},$$

the functions $\varphi(t)$, $\varphi_j(t)$, $\gamma_0(t)$ are solutions of Fred-

Fredholm

$$T_+\varphi \equiv \varphi(t) + \frac{1}{2\pi i}\int_L \left[\frac{\alpha'(\tau)}{\alpha(\tau)-\alpha(t)} - \frac{\overline{\tau'^2(\sigma)}}{\bar{\tau}-t}\right]\varphi(\tau)\,d\tau = \frac{g_+(t)}{\alpha^n(t)\chi_0^+[\alpha(t)]},$$

$$T_+\varphi_{2k-1} = \frac{1}{\bar{t}^k} - \frac{1}{\alpha^k(t)}, \qquad T_+\varphi_{2k} = -\frac{i}{\bar{t}^k} - \frac{i}{\alpha^k(t)},$$

$$T_+\gamma = \ln \bar{t}^n\alpha^{-n}(t)G(t), \qquad k = 1, 2, \dots, n,$$

$T_+\varphi$ is an operator without eigenfunctions.

For $\varkappa < 0$, problem (8) has a unique analytic solution, represented by formula (11), where one must set $\beta_1 = \beta_2 = \dots = \beta_{2n} = 0$, if and only if the $-\varkappa - 1$ solvability conditions are satisfied

$$\operatorname{Im}\frac{1}{2\pi i}\int_L \tau^{-1}\varphi[\alpha(\tau)]\,d\tau = 0, \qquad \operatorname{Re}\int_L \tau^{-k-1}\varphi[\alpha(\tau)]\,d\tau = 0,$$

$$\operatorname{Im}\int_L \tau^{-k-1}\varphi[\alpha(\tau)]\,d\tau = 0, \qquad k = 1, 2, \dots, n-1. \tag{12}$$

For $\varkappa = -2n < 0$, boundary-value problem (9) is unconditionally solvable. The general solution of this problem, vanishing at infinity, depends linearly on $-\varkappa-1$ arbitrary real constants and is expressed by the formula

$$\Phi^-(z) = z^{-n}X_0^-(z)\left[\frac{1}{2\pi i}\int_L \frac{\varphi[\alpha(\tau)]}{\tau - z}\,d\tau + \sum_{j=0}^{2n-2}\beta_j w_j^-(z) + \frac{ib}{2}\right], \tag{13}$$

$$w_0^-(z) = 1, \qquad w_{2k-1}^-(z) = z^k + \frac{1}{2\pi i}\int_L \frac{\varphi_{2k-1}[\alpha(\tau)]}{\tau - z}\,d\tau + \frac{ib_{2k-1}}{2},$$

$$w_{2k}^-(z) = iz^k + \frac{1}{2\pi i}\int_L \frac{\varphi_{2k}[\alpha(\tau)]}{\tau - z}\,d\tau + \frac{ib_{2k}}{2},$$

$$X_0^-(z) = \exp\left\{\frac{1}{2\pi i}\int_L \frac{\gamma_0[\alpha(\tau)]}{\tau - z}\,d\tau\right\}, \qquad k = 1, 2, \dots, n-1,$$

the functions $\varphi(t)$, $\varphi_j(t)$, $\gamma_0(t)$ being solutions of the equations

$$T_-\varphi \equiv -\varphi(t) + \frac{1}{2\pi i}\int_L \left[\frac{\alpha'(\tau)}{\alpha(\tau)-\alpha(t)} - \frac{\overline{\tau'^2}}{\bar{\tau}-\bar{t}}\right]\varphi(\tau)\,d\tau = \frac{g(t)}{\alpha^n(t)\chi_0^-[\alpha(t)]} - ib,$$

$$T_{-}\varphi_{2k-1} = -\alpha^k(t) + \bar{t}^k - ib_{2k-1}, \qquad T_{-}\varphi_{2k} = -i\alpha^k(t) - i\bar{t}^k - ib_{2k},$$

$T_{-}\gamma_0 = \ln \bar{t}^{-n}\alpha^n(t)G(t)$, $T_{-}\varphi$ is an operator with one eigenfunction $\varphi(t) \equiv 1$.

For $\varkappa > 0$, problem (9) is solvable and has a unique solution, obtained from (13) with $\beta_0 = \beta_1 = \dots = \beta_{2n-2} = 0$, if and only if the $\varkappa + 1$ real conditions are satisfied

$$b = \operatorname{Im} \int_L \frac{g(t)}{\alpha^n(t)\chi_0^{-}[\alpha(t)]} \psi(t)\, dt = 0,$$

$$\operatorname{Re} \int_L \varphi[\alpha(t)] t^{k-1}\, dt = 0, \qquad \operatorname{Im} \int_L \varphi[\alpha(t)] t^{k-1}\, dt = 0, \qquad k = 1, 2, \dots, n, \quad (14)$$

where $\psi(t)$ is a nontrivial solution of the equation $T'_{-}\psi = 0$, adjoint to the equation $T_{-}\varphi = 0$, and moreover

$$\int_L \psi(t)\, dt = 1.$$

In the case $\alpha(t) \equiv t$, the solutions of boundary-value problems (8) and (9) can be obtained by the same formulas (11) and (13), replacing $\alpha(t)$ in them by t, with the difference that

for an odd index $G(t)$ ($\chi = 2n - 1$) the functions $\varphi(t)$ and $\varphi_\nu(t)$ (for problem (8)) are solutions of the integral equations

$$N_{+}\varphi \equiv \varphi(t) + \frac{1}{2\pi i} \int_L \left[\frac{1}{\tau - t} - \frac{|t|\tau}{t|\tau|} \frac{\overline{\tau^2}}{\tau - t} \right] \varphi(\tau)\, d\tau = \frac{g_{+}(t)}{t^n \chi_0^{+}(t)},$$

$$N_{+}\varphi_{2k-1} = -\frac{1}{t^k} + \frac{|t|}{t\bar{t}^k}, \qquad N_{+}\varphi_{2k} = -\frac{i}{t^k} - \frac{i|t|}{t\bar{t}^k}, \qquad k = 1, 2, \dots, n.$$

An analogous result will be obtained in the corresponding case for problem (9). Both Fredholm operators $N_{+}\varphi$ and

$$N_{-}\varphi \equiv -\varphi(t) + \frac{1}{2\pi i} \int_L \left[\frac{1}{\tau - t} - \frac{|t|\tau}{t|\tau|} \frac{\overline{\tau^2}}{\tau - t} \right] \varphi(\tau)\, d\tau$$

have no eigenfunctions.

On the basis of the results of the investigation of the boundary-value problems (8) and (9) for the integral equations (1) and (2), we obtain the following conclusions:

Theorem 1. *The homogeneous equation (1) is solvable for any index* χ *and has* $\chi+1$ *linearly independent solutions if* $\chi \geq 0$*, and* $-\chi-1$ *linearly independent solutions if* $\chi < 0$*. The nonhomogeneous equation (1), generally speaking, is not solvable. For the existence of a solution of this equation it is necessary and sufficient that* $\chi+1$ *solvability conditions (14) be satisfied for* $\chi \geq 0$*, and* $-\chi-1$ *conditions (12) for* $\chi < 0$*.*

Analogous assertions are valid for the integral equation (2) for any χ, except $\chi = -1$. In this last case the homogeneous equation (2) has no solutions other than the trivial one, while the nonhomogeneous equation (2) is unconditionally solvable and has a unique solution.

The integral equation adjoint to (1) has the form*

$$\overline{t^{1/2}a[\alpha(t)]}\,\omega(t) - \frac{\alpha'(t)\,\overline{b[\alpha(t)]}}{\pi i}\int_L \frac{\omega(\tau)}{\tau-\alpha(t)}\,d\tau = 0. \tag{15}$$

Denote by χ_* the index of the coefficient of the boundary-value problem corresponding to equation (15). It is easy to see that $\chi_* = -\chi - 2$.

Comparing the numbers of linearly independent solutions of the homogeneous equations (1) and (15), we obtain that these numbers are equal to each other. The number of solvability conditions for the nonhomogeneous integral equation (1) and the number of eigenfunctions of equation (15) also coincide. Analogous conclusions are obtained for the integral equation (2). Consequently, the following are valid:

Theorem 2. *The indices of the integral equations (1) and (2) are equal to zero.*

Theorem 3. *The integral equations (1) and (2) are normally solvable.*

Thus, for the singular equations (1) and (2) the Fredholm theorems are valid. We note that the first Fredholm theorem is realized only for the integral equation (2) in the single case when $\chi = -1$.

If the contour L is the unit circle, $\alpha(t) = \dfrac{t-z_0}{\bar{z}_0 t - 1}$, $|z_0| < 1$ (or $\alpha(t) \equiv t$), then the integral equations (1) (or (2)) are solved in closed form.

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* The integral equation adjoint to equation (2) is obtained from (15) by replacing $\alpha(t)$ by t.

Note: Figure translations are in progress. See original paper for figures.