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Abstract

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MATHEMATICS

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ON THE THEORY OF A HYPERSURFACE IN FOUR-DIMENSIONAL NON-EUCLIDEAN SPACES

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1°. In the present paper, a four-dimensional non-Euclidean space of index l, ${}^{l}S_4$, is treated as a four-dimensional projective space in which there is given a non-degenerate relative symmetric metric tensor $g_{\alpha\beta}$ $(\alpha, \beta = 0, 1, 2, 3, 4)$, generating the relatively invariant quadratic form

$$g_{\alpha\beta}x^{\alpha}x^{\beta}, \tag{1}$$

whose signature has l minuses.

The invariant hyperquadric Q_3, defined by the equation

$$g_{\alpha\beta}x^{\alpha}x^{\beta} = 0, \tag{2}$$

is called the absolute of the space ${}^{l}S_4$.

With the aid of this absolute one introduces the Cayley-Klein metric ([1]). We refer the non-Euclidean space ${}^{l}S_4$ under consideration to a moving local frame consisting of five points A_{α}. The equations of infinitesimal displacement of this frame will be

$$dA_{\alpha} = \omega_{\alpha}^{\beta}A_{\beta}, \tag{3}$$

and the conditions of their complete integrability (structure equations), in the notation of exterior forms, have the form ([2])

$$D\omega_{\alpha}^{\beta} = [\omega_{\alpha}^{\gamma}\omega_{\gamma}^{\beta}] \qquad (\alpha, \beta, \gamma = 0, 1, 2, 3, 4), \tag{4}$$

where the linear differential forms ω_α^β are connected by the following 15 linear relations:

$$\omega_\alpha^\gamma g_{\gamma\beta} + \omega_\beta^\gamma g_{\alpha\gamma} = dg_{\alpha\beta}. \tag{5}$$

In what follows we shall denote spaces ${}^lS_4$ of positive curvature by ${}^lS_4^+$, and spaces ${}^lS_4$ of negative curvature by ${}^lS_4^-$.

2°. Consider in the four-dimensional non-Euclidean space a hypersurface (A), determined by the coordinates of its current point A, given as functions of three parameters u^χ $(\chi = 1, 2, 3)$. We refer the hypersurface (A) to a moving frame, identifying the point A_0 of the frame with the current point A of the hypersurface. Then the differential equations of the hypersurface have the form [3]

$$\omega_0^p = \lambda_{0\chi}^p \, du^\chi \qquad (p = 1, 2, 3, 4;\ \chi = 1, 2, 3). \tag{6}$$

Continuing this system of Pfaff equations, we obtain

$$d\lambda_{0\chi}^p - \lambda_{0\chi}^p \omega_0^0 + \lambda_{0\chi}^q \omega_q^p = \lambda_{0\chi\chi_1}^p du^{\chi_1} \qquad (\chi, \chi_1 = 1, 2, 3;\ p, q = 1, 2, 3, 4). \tag{7}$$

The quantities $g_{\alpha\beta}$, $\lambda_{0\chi}^p$ are components of the fundamental geometric object of the hypersurface of first order; the quantities $g_{\alpha\beta}$, $\lambda_{0\chi}^p$, and $\lambda_{0\chi\chi_1}^p$ serve as components of the fundamental object of second order.

Theorem. *The fundamental object of second order for a hypersurface is complete: specifying the field of the fundamental object of second order (i.e., the values of its components at each point of the domain under consideration in the parameter space) determines the hypersurface up to a non-Euclidean motion.*

3°. The following invariants and invariant geometric images are associated with the differential neighborhood of first order of the hypersurface:

a) The linear element of the hypersurface ds^2 is the square of the distance from a point A to an infinitely near point $A + dA$ of the hypersurface. In the autopolar frame of first order it has the form

$$ds^2 = \frac{r^2}{g_{00}} \{g_{11}(\omega^1)^2 + g_{22}(\omega^2)^2 + g_{33}(\omega^3)^2\}, \tag{8}$$

where r is the radius of curvature of the space.

b) The point N is the pole of the tangent hyperplane, and the line $[AN]$, joining this point with the current point A of the hypersurface. This line $[AN]$ is called the **normal** of the hypersurface. The totality of the normals of the hypersurface forms the **normal congruence** of the hypersurface.

The differential neighborhood of second order possesses one absolutely invariant quadratic differential form $\overset{2}{\Phi}$—the doubled distance of the point $A + dA + \frac{1}{2}d^2A$ of the differential neighborhood of second order from the tangent hyperplane of the hypersurface. In the autopolar frame of first order it has the form

$$\overset{2}{\Phi} = \frac{r\sqrt{g_{44}}}{\sqrt{g_{00}}}(\omega^1\omega_1^4 + \omega^2\omega_2^4 + \omega^3\omega_3^4). \tag{9}$$

As is known, two quadratic forms

$$ds^2 = a_{\chi\chi_1}du^\chi du^{\chi_1}, \qquad \overset{2}{\Phi} = b_{\chi\chi_1}du^\chi du^{\chi_1}$$

possess three algebraic invariants:

$$H = \frac{b_{\chi\chi_1}\tilde{a}^{\chi\chi_1}}{a}; \qquad I = \frac{a_{\chi\chi_1}\tilde{b}^{\chi\chi_1}}{a}, \qquad K = \frac{b}{a} \quad (a = \mathrm{Det}\,|a_{\chi\chi_1}|,\ b = \mathrm{Det}\,|b_{\chi\chi_1}|), \tag{10}$$

where $\tilde{a}^{\chi\chi_1}$ and $\tilde{b}^{\chi\chi_1}$ are the algebraic complements of the elements $a_{\chi\chi_1}$ and $b_{\chi\chi_1}$ in the determinants a and b.

With the differential neighborhood of second order there are associated three invariant directions $du^1 : du^2 : du^3$, which determine the developable surfaces of the normal congruence, and three points F_1, F_2, F_3 on the normal—the foci of the normal congruence, which describe the focal hypersurfaces.

In the Riemann space S_4 and Lobachevsky space ${}^1S_4^-$, as well as in the isometric spaces ${}^5S_4$ and ${}^4S_4^+$, all three families of developable surfaces and all three focal hypersurfaces are real.

In the spaces ${}^1S_4^+$ and ${}^2S_4^-$, in that region of the hypersurface in which the index of the tangent hyperplane is one less than the index l of the space—

the same conclusion holds; in that region of the hypersurface in which the index of the tangent hyperplane is equal to the index of the space, the normal congruence may have either one real focal hypersurface and one real family of developable surfaces, or all three. In the spaces ${}^2S_4^+$ and ${}^3S_4^-$, the normal congruence may have either one or three real focal hypersurfaces and families of developable surfaces. In the spaces ${}^3S_4^+$ and ${}^4S_4^-$, in that region of the hypersurface in which the index of the tangent hyperplane is equal to the index of the space, all three focal hypersurfaces and all three families of developable surfaces of the normal congruence are real; in the same region where the index of the tangent hyperplane is one less than the index of the space, the normal congruence may have either one real family of developable surfaces and one real focal hypersurface, or all three.

In the spaces ${}^1S_4^+$, ${}^2S_4^-$, ${}^2S_4^+$, ${}^3S_4^+$, ${}^3S_4^-$, ${}^4S_4^-$, the tangent hyperplane of a hypersurface may be isotropic.

Theorem. *If, for a hypersurface (A), at every point the tangent hyperplane is isotropic, then this hypersurface is a lineation hypersurface of rank two with isotropic rectilinear generators.*

4°. Consider a curve $u^\alpha = f^\alpha(t)$ on the hypersurface (A). The ratio of the doubled distance of the point $A + dA + \frac{1}{2}d^2A$, under displacement along this curve, to the tangent hyperplane, to the linear element gives the normal curvature of this curve

$$K_n = \frac{\overset{2}{\Phi}}{ds^2}. \tag{11}$$

Stationary values of the normal curvature are called **principal curvatures**, and the directions in which they are realized are called **principal directions**. Lines tangent at each of their points to principal directions are called **lines of curvature**.

Denote the three principal curvatures of the hypersurface by k_1, k_2, k_3. The invariants (10) are the elementary symmetric polynomials in the principal curvatures of the family

$$H = k_1 + k_2 + k_3, \qquad I = k_1k_2 + k_1k_3 + k_2k_3, \qquad K = k_1k_2k_3. \tag{12}$$

For the normal curvature of a curve t on a hypersurface, Euler' s formula holds:

$$k_n = k_1 \cos^2\varphi_1 + k_2 \cos^2\varphi_2 + k_3 \cos^2\varphi_3, \tag{13}$$

where $\varphi_1, \varphi_2, \varphi_3$ are the angles formed by the curve t with the principal directions.

The developable surfaces of the normal congruence cut the hypersurface along lines of curvature.

From this we obtain that a hypersurface in elliptic space S_4, in Lobachevsky space ${}^1S_4^-$, and in spaces isometric to them ${}^5S_4$ and ${}^4S_4^+$, always carries three real families of lines of curvature. In the spaces ${}^1S_4^+$, ${}^2S_4^-$, ${}^3S_4^+$, and ${}^4S_4^-$, in that region of the hypersurface in which the tangent hyperplanes have respectively index $0, 1, 3, 4$, all three families of lines of curvature will be real, while in that region of the hypersurface in which the tangent hyperplanes have respectively index $1, 2, 2, 3$, there may be either three real families of lines of curvature or one (the remaining two will be imaginary conjugates). In the spaces ${}^2S_4^+$ and ${}^3S_4^-$, a hypersurface may carry either three real families of lines of curvature, or one real family and two imaginary conjugate ones.

Between the distance ρ of the focus F of the normal from the point A of the hypersurface and the corresponding principal curvature k_x there is the relation

$$\operatorname{ctg}\frac{\rho}{r} = kr. \tag{14}$$

In that region of the hypersurface in which the index of the tangent hyperplane is equal to the index of the space, a real principal direction has a real principal curvature, while imaginary conjugate principal directions have imaginary conjugate principal curvatures. In that region of the hypersurface in which the index of the tangent hyperplane is one less than the index of the space, a real principal direction has a purely imaginary curvature, while imaginary conjugate principal directions have imaginary (but not conjugate) principal curvatures.

At those points of the hypersurface at which all three principal curvatures are distinct, the principal directions are determined and mutually orthogonal.

If at every point of a hypersurface in the spaces S_4, ${}^1S_4^-$, ${}^4S_4^+$ and ${}^5S_4$ two principal curvatures are equal, then the hypersurface is foliated into ∞^1 two-dimensional non-Euclidean spheres.

If at every point of a hypersurface in the spaces ${}^1S_4^+$, ${}^2S_4^-$, ${}^2S_4^+$, ${}^3S_4^-$, ${}^3S_4^+$, ${}^4S_4^-$ two principal curvatures are equal, then either the hypersurface is foliated into ∞^1 two-dimensional spheres (if the tangent planes to these two principal directions are both either elliptic or hyperbolic), or the hypersurface is foliated into ∞^1 two-dimensional developable surfaces whose rectilinear generators are isotropic (this will occur if, of the tangent planes to the two principal directions, one is elliptic and the other hyperbolic).

If at every point of a hypersurface in the spaces S_4, ${}^1S_4^-$, ${}^4S_4^+$, ${}^5S_4$ all three principal curvatures are equal, then it is a hypersphere. If at every point of a hypersurface in the spaces ${}^1S_4^+$, ${}^2S_4^-$, ${}^2S_4^+$, ${}^3S_4^-$, ${}^3S_4^+$, ${}^4S_4^-$ all three principal curvatures are equal, then either it is a hypersphere, or the hypersurface is foliated into ∞^1 two-dimensional developable surfaces with isotropic rectilinear generators.

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